# THE ZERO FORCING NUMBER OF BIPARTITE GRAPHS

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## Abstract

Graph theory is the study of certain mathematical objects called graphs. A *graph* consists of points, called *vertices*, which are connected by line segments or arcs called *edges*. Graphs are commonly used to represent situations found in real life. Airports and flight patterns, computers and network connections, and towns and roadways are some examples of real life situations where graph theory is applied. Graph coloring is an established area of research that is used in applications of graph theory. In this project we color the vertices of a graph either black or white. We use the following *basic color change rule*: If a white vertex is the only white vertex adjacent to a black vertex, then we change the color of that white vertex to black. We start by coloring some vertices white and some vertices black. The goal is to find the minimum number of black vertices so that all white vertices are changed to black by a repeated application of the basic color change rule.

The zero forcing number of a graph $G$, $Z(G)$, is the minimum number of vertices to be colored black at the start so that all vertices become black by repeated use of the basic color change rule. There is a related notion called the semidefinite zero forcing number where the semidefinite color change rule is slightly different from the basic color change rule. In this project we find the zero forcing number and semidefinite zero forcing number of an important class of graphs called bipartite graphs.

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## 1. Introduction

Graph theory is the study of certain mathematical objects called graphs. A *graph* consists of points, called *vertices*, which can be connected by line segments or arcs called *edges*. They are often used to represent real life situations, such as airports and flight patterns, computers and network connections, and towns and roadways. A bipartite graph, also referred to as a bigraph, is a graph whose vertex set is decomposed into two disjoint sets such that no two vertices within the same set are adjacent. In the following example of a bipartite graph (Figure 1.1) we could think of vertices A, B, and C as representing candidates applying for jobs, and 1, 2, 3, 4, 5, and 6 as representing jobs in a company. Two vertices are joined by an edge if the candidate has the qualification for a particular job. Note that there can be no edges between vertices A, B, and C or between vertices 1, 2, 3, 4, 5, and 6.

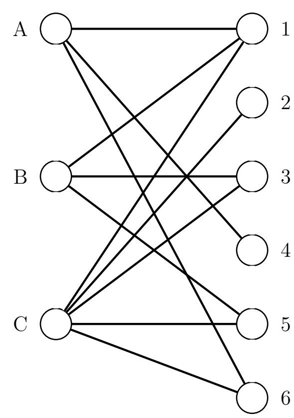



Figure 1.1

One well established area of graph theory is graph coloring [12]. In graph coloring, a common method is to look for the minimum number of colors used to color every vertex of a graph so that any two adjacent vertices do not have the same color. The following is an application of graph coloring: Suppose you want to build compartments to store chemicals in a room. However, some of these chemicals can react with each other, and hence should be stored in separate compartments. Suppose there are 6 chemicals. We denote them as vertices 1, 2, 3, 4, 5, and 6 in a graph (Figure 2.1). Two vertices are joined by an edge if the chemicals react with each other.

Figure 2.1

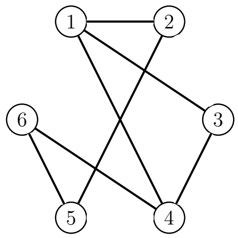
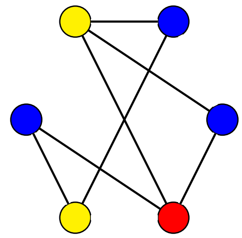



Figure 2.2



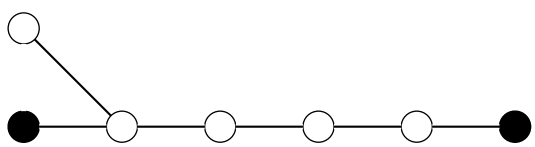
The second graph (Figure 2.2) shows a possible minimum coloring of the graph with colors yellow (Y), blue (B), and red (R). Since three colors are needed, it is possible that the chemicals can be stored in three compartments without chemicals reacting with one another.

In this research we color the vertices with only two colors, black and white. We use two different color change rules which are described below.

1.1. **Basic Color Change Rule.** The *basic color change rule* states that if a white vertex is the only white vertex adjacent to a black vertex, then we change the color of that white vertex to black [1].

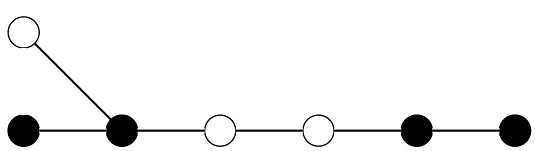
For example, we first color the vertices of a graph either black or white (Figure 3.1).

Figure 3.1



The coloring of the graph in Figure 3.1 would be changed to the coloring in Figure 3.2 using the basic color change rule. There are two white vertices which are the only white vertices adjacent to black vertices, so we change their colors to black (Figure 3.2).

Figure 3.2



By repeated applications of the basic color change rule, we are able to change all vertices to black. The zero forcing number of a graph $G$, $Z(G)$, is the minimum number of vertices to be colored black at the start so that all vertices will become black by repeated use of the basic color change rule. Since $Z(G)$ can be seen to be greater than 1, we have shown that $Z(G) = 2$ for the graph in Figure 3.1.

1.2. **Semidefinite Color Change Rule.** In the *semidefinite color change rule*, we remove all vertices that are colored black, which may disconnect the graph into components. Then we add the black vertices back to each component, and use the basic color change rule [2].

For example, the following graph in Figure 4.1 is broken up into two separate components when the single black vertex is removed, as seen in Figure 4.2.

Figure 4.1

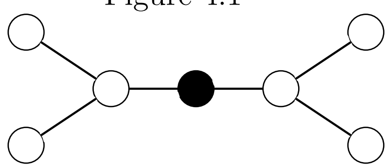
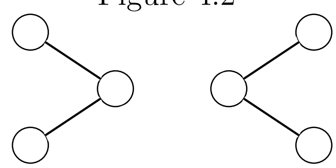


Figure 4.2



Then the black vertex is added back to each component (Figure 4.3), and the basic color change rule is applied to get the following coloring (Figure 4.4).

Figure 4.3

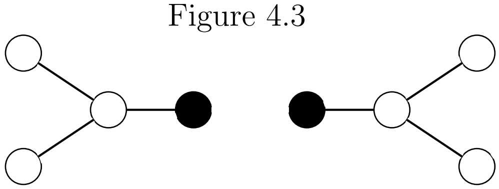
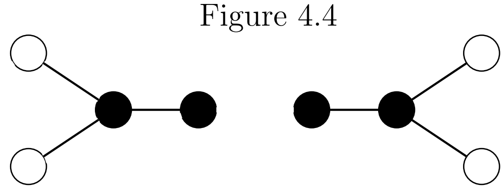


Figure 4.4



This step results in three black vertices in the original graph (Figure 4.5).

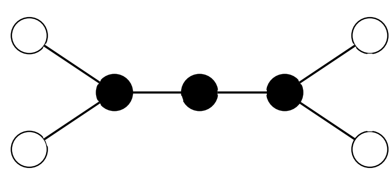


Figure 4.5

We repeat the process by removing these three black vertices. By repeated application of the semidefinite color change rule, we will be able to change all of the vertices to black. The semidefinite zero forcing number of a graph $G$, $Z_+(G)$, is the minimum number of vertices to be colored black at the start so that all the vertices will become black by repeated use of the semidefinite color change rule. Note that $Z_+(G) = 1$ for the graph in Figure 4.1, but it is easy to observe that $Z(G)$ is greater than 1 (Figure 3.1). Therefore, in general, $Z_+(G) \leq Z(G)$.

## 2. Literature Review

The zero forcing number for graphs was first introduced in [1] by the AIM Minimum Rank - Special Graphs Work Group, which included Dr. Narayan. The semidefinite zero forcing number for graphs was introduced in [2]. Research on the minimum rank of graphs can be found in [9] and [13]. The minimum semidefinite rank of graphs was further explored in [3, 4, 6-8, 10, 13-16], where the minimum semidefinite rank was calculated for various families of graphs, including tree graphs such as the ones in Figures 1.1, 2.1, 3.1 and 4.1 for which the semidefinite zero forcing number was shown to be 1. We will be expanding on this research by looking into new types of graphs, in particular, bipartite graphs.

## 3. Results

### 3.1. **Zero Forcing Number.**

We start by finding the zero forcing number of a complete bipartite graph, $K_{m,n}$. This is a graph on $m+n$ vertices. The two partite sets, X and Y, have coordinates m and n respectively. Every vertex in X is joined to every vertex in Y to form $K_{m,n}$.

*Proposition* 1. Let $K_{m,n}$ be a complete bipartite graph. Then $Z(K_{m,n}) = m+n-2$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. If we color all vertices except $x_1$ and $y_1$ black, then by the basic color change rule $x_2$ will force $y_1$ black because that is the only white vertex adjacent to $x_2$ in $Y$. Similarly, $y_2$ will force $x_1$ black. Hence, $Z(K_{m,n}) \leq m+n-2$.

To show $Z(K_{m,n})$ is not less than $m+n-2$, we consider coloring all but three vertices of $K_{m,n}$ black.

Case 1 Suppose any three vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all three of the white vertices in $X$ black.

Case 2 Suppose any two vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force both of the white vertices in $X$ black.

Since the other (two) cases are similar we can conclude that $Z(K_{m,n}) > m+n-3$. Hence, $Z(K_{m,n}) = m + n - 2$. □

Now we find the zero forcing number of graphs where certain edges are removed from the complete bipartite graph, $K_{m,n}$.

*Proposition* 2. Let $K_{m,n} - e$ be a complete bipartite graph with one edge removed. Then $Z(K_{m,n} - e) = m + n - 3$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. Assume that the graph $K_{m,n} - e$ is obtained from $K_{m,n}$ by removing the edge $e$ from $x_1$ to $y_1$. If we color all vertices except $x_2$, $y_1$, and $y_2$ black, then by the basic color change rule $x_1$ will force $y_2$ black because that is the only white vertex adjacent to $x_1$ in $Y$. Now, $y_2$ will force $x_2$ black and then $x_2$ will force $y_1$ black. Hence, $Z(K_{m,n} - e) \leq m + n - 3$.

To show $Z(K_{m,n} - e)$ is not less than $m + n - 3$, we consider coloring all but four vertices of $K_{m,n} - e$ black.

Case 1 Suppose any four vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all four of the white vertices in $X$ black.

Case 2 Suppose any three vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all three of the white vertices in $X$ black.

Case 3 Suppose all but two vertices in $X$ and all but two vertices in $Y$ are colored black. By the basic color change rule, the black vertices in $Y$ cannot force both of the white vertices in $X$ black and the black vertices in $X$ cannot force both of the white vertices in $Y$ black.

Since the other (two) cases are similar we can conclude that $Z(K_{m,n} - e) > m + n - 4$. Hence, $Z(K_{m,n} - e) = m + n - 3$. □

*Proposition* 3. Let $K_{m,n}-2e$ be a complete bipartite graph with two edges removed. Suppose the two edges removed are incident on a single vertex. Then $Z(K_{m,n} - 2e) = m + n - 3$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. Assume that the graph $K_{m,n} - 2e$ is obtained from $K_{m,n}$ by removing two edges incident on a single vertex. Suppose edges $x_1y_1$ and $x_1y_2$ are removed from $K_{m,n}$. If we color all vertices except $x_2$, $y_1$, and $y_3$ black, then by the basic color change rule $x_1$ will force $y_3$ black because that is the only white vertex adjacent to $x_1$. Now, $x_3$ will force $y_1$ black and $y_2$ will force $x_2$ black. Hence, $Z(K_{m,n} - 2e) \leq m + n - 3$.

To show $Z(K_{m,n} - 2e)$ is not less than $m+n-3$, we consider coloring all but four vertices of $K_{m,n} - 2e$ black.

Case 1 Suppose any four vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all four of the white vertices in $X$ black.

Case 2 Suppose any three vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all three of the white vertices in $X$ black.

Case 3 Suppose all but two vertices in $X$ and all but two vertices in $Y$ are colored black. By the basic color change rule, the black vertices in $Y$ cannot force both of the white vertices in $X$ black and the black vertices in $X$ cannot force both of the white vertices in $Y$ black.

Since the other (two) cases are similar we can conclude that $Z(K_{m,n} - 2e) > m + n - 4$. Hence, $Z(K_{m,n} - 2e) = m + n - 3$. □

*Proposition* 4. Let $K_{m,n} - 2e$ be a complete bipartite graph with two edges removed. Suppose the two edges removed have no common vertices. Then $Z(K_{m,n} - 2e) = m + n - 4$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. Assume that the graph $K_{m,n} - 2e$ is obtained from $K_{m,n}$ by removing two edges having no common vertices. Suppose edges $x_1y_1$ and $x_2y_2$ are removed from $K_{m,n}$. If we color all vertices except $x_2$, $x_3$, $y_1$,and $y_2$ black, then by the basic color change rule $x_1$ will force $y_2$ black because that is the only white vertex adjacent to $x_1$. Then, $y_2$ will force $x_3$ black, $x_4$ will force $y_1$, and $y_3$ will force $x_2$ black. Hence, $Z(K_{m,n} - 2e) \leq m + n - 4$.

To show $Z(K_{m,n} - 2e)$ is not less than $m + n - 4$, we consider coloring all but five vertices of $K_{m,n} - 2e$ black.

<u>Case 1</u> Suppose any five vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all five of the white vertices in $X$ black.

<u>Case 2</u> Suppose any four vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all four of the white vertices in $X$ black.

<u>Case 3</u> Suppose all but three vertices in $X$ and all but two vertices in $Y$ are colored black. By the basic color change rule, the black vertices in $Y$ cannot force all three of the white vertices in $X$ black and the black vertices in $X$ cannot force both of the white vertices in $Y$ black.

Since the other (three) cases are similar we can conclude that $Z(K_{m,n} - 2e) > m + n - 5$. Hence, $Z(K_{m,n} - 2e) = m + n - 4$. □

*Proposition* 5. Let $K_{m,n} - 3e$ be a complete bipartite graph with three edges removed. Suppose the three edges removed are incident on a single vertex. Then $Z(K_{m,n} - 3e) = m + n - 3$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. Assume that the graph $K_{m,n} - 3e$ is obtained from $K_{m,n}$ by removing three edges incident on a single vertex. Suppose edges $x_1y_1$, $x_1y_2$, and $x_1y_3$ are removed from $K_{m,n}$. If we color all vertices except $x_2$, $y_1$, and $y_4$ black, then by the basic color change rule $x_1$ will force $y_4$ black because that is the only white vertex adjacent to $x_1$. Now, $x_3$ will force $y_1$ black and $y_2$ will force $x_2$ black. Hence, $Z(K_{m,n} - 3e) \leq m + n - 3$.

To show $Z(K_{m,n} - 3e)$ is not less than $m + n - 3$, we consider coloring all but four vertices of $K_{m,n} - 3e$ black.

<u>Case 1</u> Suppose any four vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all four of the white vertices in $X$ black.

<u>Case 2</u> Suppose any three vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all three of the white vertices in $X$ black.

<u>Case 3</u> Suppose all but two vertices in $X$ and all but two vertices in $Y$ are colored black. By the basic color change rule, the black vertices in $Y$ cannot force both of the white vertices in $X$ black and the black vertices in $X$ cannot force both of the white vertices in $Y$ black.

Since the other (two) cases are similar we can conclude that $Z(K_{m,n} - 3e) > m + n - 4$. Hence, $Z(K_{m,n} - 3e) = m + n - 3$. □

*Proposition* 6. Let $K_{m,n} - 3e$ be a complete bipartite graph with three edges removed. Suppose the two of the edges removed are incident on a single vertex and one edge is incident on a different vertex are removed. Then $Z(K_{m,n} - 3e) = m + n - 4$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. Assume that the graph $K_{m,n} - 3e$ is obtained from $K_{m,n}$ by removing two edges incident on a single vertex and one edge incident on a different vertex. There are two possible cases.

<u>Case 1</u> Suppose edges $x_1y_1$, $x_1y_2$, and $x_2y_3$ are removed from $K_{m,n}$. If we color all vertices except $x_2$, $x_3$, $y_1$,and $y_3$ black, then by the basic color change rule $x_1$ will force $y_3$ black because that is the only white vertex adjacent to $x_1$. Then, $y_3$ will force $x_3$ black, then $x_3$ will force $y_1$, and then $y_1$ will force $x_2$ black. Hence, $Z(K_{m,n} - 3e) \leq m + n - 4$.

<u>Case 2</u> Suppose edges $x_1y_1$, $x_1y_2$, and $x_2y_1$ are removed from $K_{m,n}$. If we color all vertices except $x_2$, $y_1$, $y_2$,and $y_3$ black, then by the basic color change rule $x_1$ will force $y_3$ black because that is the only white vertex adjacent to $x_1$. Then, $y_3$ will force $x_2$ black, then $x_2$ will force $y_2$, and then $x_3$ will force $y_1$ black. Hence, $Z(K_{m,n} - 3e) \leq m + n - 4$.

To show $Z(K_{m,n} - 3e)$ is not less than $m+n-4$, we consider coloring all but five vertices of $K_{m,n} - 3e$ black.

<u>Case 1</u> Suppose any five vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all five of the white vertices in $X$ black.

<u>Case 2</u> Suppose any four vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all four of the white vertices in $X$ black.

<u>Case 3</u> Suppose all but three vertices in $X$ and all but two vertices in $Y$ are colored black. Black vertices in $X$ will force the two white vertices in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all three of the white vertices in $X$ black.

Since the other (three) cases are similar we can conclude that $Z(K_{m,n} - 3e) > m + n - 5$. Hence, $Z(K_{m,n} - 3e) = m + n - 4$. □

*Proposition* 7. Let $K_{m,n} - 3e$ be a complete bipartite graph with three edges removed. Suppose the three edges removed have no common vertices. Then $Z(K_{m,n} - 3e) = m + n - 4$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. Assume that the graph $K_{m,n} - 3e$ is obtained from $K_{m,n}$ by removing three edges having no common vertices. Suppose edges $x_1y_1$, $x_2y_2$, and $x_3y_3$ are removed from $K_{m,n}$. If we color all vertices except $x_2$, $x_3$, $y_1$,and $y_2$ black, then by the basic color change rule $x_1$ will force $y_2$ black because that is the only white vertex adjacent to $x_1$. Then, $y_2$ will force $x_3$ black, $x_4$ will force $y_1$, and $y_3$ will force $x_2$ black. Hence, $Z(K_{m,n} - 3e) \leq m + n - 4$.

To show $Z(K_{m,n} - 3e)$ is not less than $m + n - 4$, we consider coloring all but five vertices of $K_{m,n} - 3e$ black.

<u>Case 1</u> Suppose any five vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all five of the white vertices in $X$ black.

<u>Case 2</u> Suppose any four vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all four of the white vertices in $X$ black.

<u>Case 3</u> Suppose all but three vertices in $X$ and all but two vertices in $Y$ are colored black. By the basic color change rule, the black vertices in $Y$ cannot force all three of the white vertices in $X$ black and the black vertices in $X$ cannot force both of the white vertices in $Y$ black.

Since the other (three) cases are similar we can conclude that $Z(K_{m,n} - 3e) > m + n - 5$. Hence, $Z(K_{m,n} - 3e) = m + n - 4$. □

*Proposition* 8. Let $K_{m,n} - re$ be a complete bipartite graph with $r$ edges removed, where $0 < r < n \leq m$. Suppose all the edges are removed incident on a single vertex. Then $Z(K_{m,n} - re) = m + n - 3$.

*Proof.* Let $X$ and $Y$ be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. We will assume without loss of generality that $m \geq n$.

Suppose we label the vertices in $X$ as $x_1$, $x_2$,..., $x_m$ and the vertices in $Y$ as $y_1$, $y_2$,..., $y_n$. Assume that the graph $K_{m,n} - re$ is obtained from $K_{m,n}$ by removing $r$ edges incident on a single vertex. Suppose edges $x_1y_1...x_1y_r$ are removed from $K_{m,n}$. If we color all vertices except $x_2$, $y_1$, and $y_n$ black, then by the basic color change rule $x_1$ will force $y_n$ black because that is the only white vertex adjacent to $x_1$. Now, $x_3$ will force $y_1$ black, and $y_2$ will force $x_2$ black. Hence, $Z(K_{m,n} - re) \leq m + n - 3$.

To show $Z(K_{m,n} - re)$ is not less than $m+n-3$, we consider coloring all but four vertices of $K_{m,n} - re$ black.

Case 1 Suppose any four vertices in $X$ are colored white and the rest of the vertices in $X$ and all vertices of $Y$ are colored black. Then by the basic color change rule, the black vertices in $Y$ cannot force all four of the white vertices in $X$ black.

Case 2 Suppose any three vertices in $X$ are colored white and the rest of the vertices in $X$ are colored black and a single vertex in $Y$ is colored white and the rest of the vertices in $Y$ are colored black. A black vertex in $X$ will force the white vertex in $Y$ black by the basic color change rule. But the black vertices in $Y$ cannot force all three of the white vertices in $X$ black.

Case 3 Suppose all but two vertices in $X$ and all but two vertices in $Y$ are colored black. By the basic color change rule, the black vertices in $Y$ cannot force both of the white vertices in $X$ black and the black vertices in $X$ cannot force both of the white vertices in $Y$ black.

Since the other (two) cases are similar we can conclude that $Z(K_{m,n} - re) > m + n - 4$. Hence, $Z(K_{m,n} - re) = m + n - 3$. □

*Observation.* Let $K_{m,n}$ be a complete bipartite graph. We will find the maximum number of edges that can be removed from $K_{m,n}$ so that the resulting graph on the $(m + n)$ vertices of $K_{m,n}$ is connected. Denote the resulting graph by $G$. Note that $G$ cannot have a cycle. In other words we are interested in finding a spanning tree of $K_{m,n}$. A spanning tree of $G$ of $K_{m,n}$ on $m+n$ vertices must have $m+n-1$ edges, since a graph H is a tree if and only if the number of edges is $|H|$ [17]. Since $K_{m,n}$ has $mn$ edges, the maximum number of edges that can be removed is $mn-(m+n-1) = mn-m-(n-1) = m(n-1)-(n-1) = (m-1)(n-1)$.

*Proposition* 9. Let $K_{m,n} - pe$ be a complete bipartite graph with $p = (m-1)(n-1)$ edges removed. Then $Z(K_{m,n} - pe)$ is 1 when the resulting graph $K_{m,n} - pe$ is a path, which happens when $m = n$ or $m = n + 1$.

*Proof.* Since $K_{m,n} - pe$, where $p = (m-1)(n-1)$ is a path, we color one of the end vertices black. This will force every other vertex of $K_{m,n} - pe$ black by the basic color change rule. Hence, $Z(K_{m,n} - pe) = 1$ when $K_{m,n} - pe$ is a path, which happens when $m > n + 1$. When $m = n$ and $p = (m-1)(n-1)$ edges are removed, the resulting graph is a path on $2n$ vertices and $2n - 1$ edges. Hence, it starts in a partite set and ends in the other partite set. When $m = n+1$ and $p = (m-1)(n-1)$ edges are removed, the resulting graph is a path on $2n+1$ vertices and $2n$ edges. The path starts and ends in the same partite set. □

*Proposition* 10. Let $K_{m,n} - pe$ be a complete bipartite graph with $p = (m-1)(n-1)$ edges removed. Then $Z(K_{m,n} - pe)$ is $m - n$ when the resulting graph $K_{m,n} - pe$ is a tree that is not a path, which happens when $m > n + 1$.

*Proof.* When $m > n+1$ and $p = (m-1)(n-1)$ edges are removed, then there are $m-n-1$ extra vertices on one partite set that are connected to a vertex $V$ of maximum degree $m-n+1$ in the other partite set. Thus, we color all pendant vertices adjacent to $V$. This will force all vertices black using the basic color change rule. Hence, $Z(K_{m,n} - pe) = m - n$. □

3.2. **Semidefinite Zero Forcing Number.** We start by finding the semidefinite zero forcing number of a complete bipartite graph $K_{m,n}$. The result can also be obtained by using the fact that $OS(G) + Z_+(G) = |G|$ [Theorem 3.6, 2] and the result that $OS(K_{m,n}) = maxm, n$ [Proposition 3.10, 7].

*Proposition* 11. Let $K_{m,n}$ be a complete bipartite graph. Then $Z_+(K_{m,n}) = min\{m, n\}$.

*Proof.* We will assume without loss of generality that $m \geq n$ and show that $Z_+(K_{m,n}) = n$.

Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\}, ...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 \leq k \leq m$ in $K_{m,n}$, we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants.

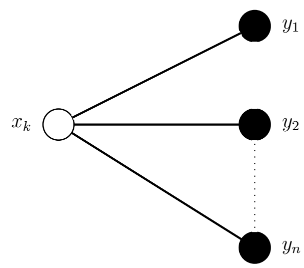


Figure 5.1

So any vertex in Y will force $x_k$ black for $1 \leq k \leq m$. Hence, $Z_+(K_{m,n}) \leq n$.

If we color only $n-1$ vertices black, then we will get a single connected component of white vertices. If we attach the set of black vertices to this component we get the induced graph to be $K_{m,n}$. But none of the black vertices can force any white vertex black. Hence, $Z_+(K_{m,n}) = n$ □

*Proposition* 12. Let $K_{m,n} - e$ be a complete bipartite graph with one edge removed. Then

$$Z_+(K_{m,n} - e) = \begin{cases} min\{\text{m,n}\} & \text{when } m > n \\ min\{\text{m,n}\} & \text{when } m = n > 2 \\ 1 & \text{when } m = n = 2 \end{cases}$$

*Proof.* Since we are considering only connected graphs, we assume $m \geq n \geq 2$.

Case 1 Let $m > n$:

Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{m,n}$. Assume that the graph $K_{m,n} - e$ is obtained from $K_{m,n}$ by removing an edge from $x_1$ to $y_1$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\}, ...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{m,n} - e$, we get a star graph with $x_1$ as the center and $y_2, ..., y_n$ as pendants.

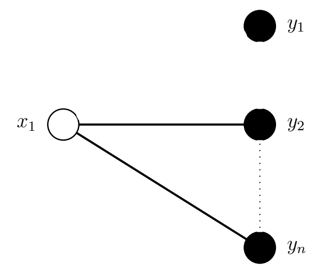



Figure 6.1

So any vertex $y_2, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as in Figure 5.1. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - e) \leq n$.

If we color only $n - 1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{m,n} - e) = n$.

Case 2 Let $m = n > 2$:

Let $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{n,n}$. Assume that the graph $K_{n,n} - e$ is obtained from $K_{n,n}$ by removing an edge from $x_1$ to $y_1$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{n,n}$, then we get the $n$ components $\{x_1\}, ...\{x_n\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{n,n} - e$, we get a star graph with $x_1$ as the center and $y_2, ..., y_n$ as pendants, as in Figure 6.1. So any vertex $y_2, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq n$ in $K_{n,n} - e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as in Figure 5.1. So any vertex in Y will force $x_k$ black for $1 < k \leq n$. Hence, $Z_+(K_{n,n} - e) \leq n$.

If we color only $n-1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{n,n}-e) = n$.

<u>Case 3</u> Let $m = n = 2$:

Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be the partite sets of $K_{2,2}$. Assume $K_{2,2}-e$ is obtained by removing one edge, $x_1y_1$, from $K_{2,2}$ If we color $x_2$ black, then removal of $x_2$ will give two components namely $\{x_1\}$ and the edge $x_1y_2$. By the semidefinite color change rule we can force $y_1$ black and $y_2$ black. Again, by the semidefinite color change rule $y_2$ will force $x_1$ black. Therefore, $Z_+(K_{2,2}-e) \leq 1$. Since at least one vertex has to be colored black, $Z_+(K_{2,2}-e) = 1$. □

*Proposition* 13. Let $K_{m,n}-2e$ be a complete bipartite graph with two edges removed. Suppose the two edges removed are incident on a single vertex. Then

$$Z_+(K_{m,n}-2e) = \begin{cases} min\{\text{m,n}\} & \text{when } m > n \\ min\{\text{m,n}\} & \text{when } m = n > 3 \\ 2 & \text{when } m = n = 3 \end{cases}$$

*Proof.* Since we are considering only connected graphs, we assume $m \geq n \geq 3$.

<u>Case 1</u> Let $m > n$:

Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{m,n}$. Assume that the graph $K_{m,n}-2e$ is obtained by removing two edges, $x_1$ to $y_1$ and $x_1$ to $y_2$, from $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\}, ...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{m,n}-2e$, we get a star graph with $x_1$ as the center and $y_3, ..., y_n$ as pendants.

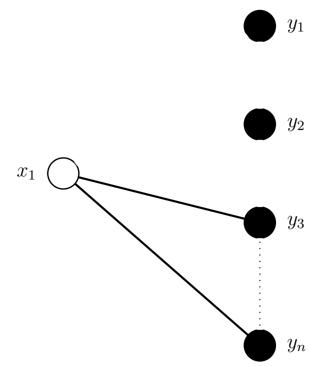



Figure 7.1

So any vertex $y_3, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - 2e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - 2e) \leq n$.

If we color only $n - 1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{m,n} - 2e) = n$.

<u>Case 2</u> Let $m = n > 3$:

Let $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{n,n}$. Assume that the graph $K_{n,n} - 2e$ is obtained by removing two edges, $x_1$ to $y_1$ and $x_1$ to $y_2$, from $K_{n,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{n,n}$, then we get the $m$ components $\{x_1\}, ...\{x_n\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{n,n} - 2e$, we get a star graph with $x_1$ as the center and $y_3, ..., y_n$ as pendants, as in Figure 7.1. So any vertex $y_3, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq n$ in $K_{n,n} - 2e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq n$. Hence, $Z_+(K_{n,n} - 2e) \leq n$.

If we color only $n-1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{n,n}-2e)=n$.

<u>Case 3</u> Let $m=n=3$:

Let $X=\{x_1,x_2,x_3\}$ and $Y=\{y_1,y_2,y_3\}$ be the partite sets of $K_{3,3}$. Assume $K_{3,3}-2e$ is obtained by removing two edges, $x_1y_1$ and $x_1y_2$, from $K_{3,3}$. If we color $x_2$ black and $x_3$ black then removal of $x_2$ and $x_3$ will give three components namely $\{y_1\}$, $\{y_2\}$, and the edge $x_1y_3$. By the semidefinite color change rule, we can force $y_1$, $y_2$, and $y_3$ black. Again by the semidefinite color change rule $y_3$ will force $x_1$ black. Therefore, $Z_+(K_{3,3}-2e)\leq 2$. If we color only one vertex black, then we will get a connected component of white vertices. If we attach the black vertex back to this component we get the induced graph to be $K_{3,3}-2e$. But the black vertex cannot force any of the white vertices black. Hence, $Z_+(K_{3,3}-2e)=3$. □

*Proposition* 14. Let $K_{m,n}-2e$ be a complete bipartite graph with two edges removed. Suppose the two edges removed have no common vertices. Then $Z_+(K_{m,n}-2e)=min\{m,n\}$.

*Proof.* We will assume without loss of generality that $m\geq n$ and show that $Z_+(K_{m,n}-2e)=n$.

Let $X=\{x_1,...,x_m\}$ and $Y=\{y_1,...,y_n\}$ be the partite sets of $K_{m,n}$. Assume that the graph $K_{m,n}-2e$ is obtained by removing two edges, $x_1$ to $y_1$ and $x_2$ to $y_2$, from $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1,...,y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\},...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y\bigcup\{x_1\}$ in $K_{m,n}-2e$, we get a star graph with $x_1$ as the center and $y_2,...,y_n$ as pendants.

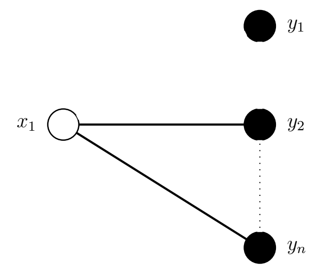



Figure 8.1

So any vertex $y_2, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_2\}$ in $K_{m,n} - 2e$, we get a star graph with $x_2$ as the center and $y_1, y_3, y_4, ..., y_n$ as pendants.

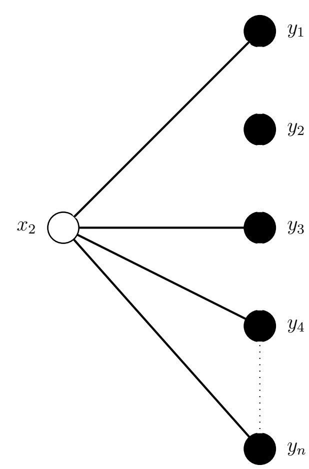



Figure 8.2

So any vertex $y_1, y_3, y_4, ..., y_n$ will force $x_2$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - 2e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - 2e) \leq n$.

If we color only $n-1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{m,n}-2e)=n$. □

*Proposition* 15. Let $K_{m,n}-3e$ be a complete bipartite graph with three edges removed. Suppose the three edges removed are incident on a single vertex. Then

$$Z_+(K_{m,n}-3e)=\begin{cases} min\{\text{m,n}\} & \text{when } m>n \\ min\{\text{m,n}\} & \text{when } m=n>4 \\ 3 & \text{when } m=n=4 \end{cases}$$

*Proof.* Since we are considering only connected graphs, we assume $m \geq n \geq 4$.

<u>Case 1</u> Let $m>n$:

Let $X=\{x_1,...,x_m\}$ and $Y=\{y_1,...,y_n\}$ be the partite sets of $K_{m,n}$. Assume that the graph $K_{m,n}-3e$ is obtained by removing three edges, $x_1$ to $y_1$, $x_1$ to $y_2$, and $x_1$ to $y_3$, from $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1,...,y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\},...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y\bigcup\{x_1\}$ in $K_{m,n}-3e$, we get a star graph with $x_1$ as the center and $y_4,...,y_n$ as pendants.

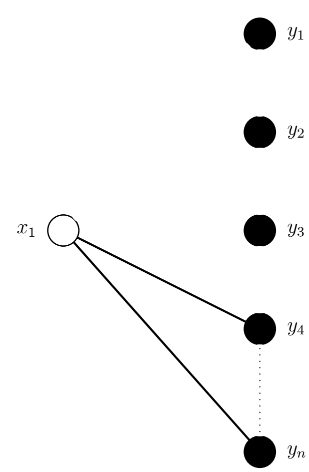



Figure 9.1

So any vertex $y_4, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - 3e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - 3e) \leq n$.

If we color only $n-1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{m,n} - 3e) = n$.

<u>Case 2</u> Let $m = n > 3$:

Let $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{n,n}$. Assume that the graph $K_{n,n} - 3e$ is obtained by removing three edges, $x_1$ to $y_1$, $x_1$ to $y_2$, and $x_1$ to $y_3$, from $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{n,n}$, then we get the $m$ components $\{x_1\}, ...\{x_n\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{n,n} - 3e$, we get a star graph with $x_1$ as the center and $y_4, ..., y_n$ as pendants, as in Figure 9.1. So any vertex $y_4, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{n,n} - 3e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq n$. Hence, $Z_+(K_{n,n} - 3e) \leq n$.

If we color only $n-1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{n,n} - 3e) = n$.

<u>Case 3</u> Let $m = n = 3$:

Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, x_4\}$ be the partite sets of $K_{4,4}$. Assume $K_{4,4} - 3e$ is obtained by removing three edges, $x_1y_1$, $x_1y_2$, and $x_1y_3$, from $K_{4,4}$. If we color $x_2$, $x_3$, and $x_4$ black, then removal of $x_2$, $x_3$, and $x_4$ will give four components namely $\{y_1\}$,

$\{y_2\}$, $\{y_3\}$ and the edge $x_1y_4$. By the semidefinite color change rule, we can force $y_1$, $y_2$, $y_3$ and $y_4$ black. Again by the semidefinite color change rule $y_4$ will force $x_1$ black. Therefore, $Z_+(K_{4,4}-3e) \leq 3$. If we color only two vertices black, then we will get a connected component of white vertices. If we attach the black vertices back to this component we get the induced graph to be $K_{4,4} - 3e$. But the black vertices cannot force any of the white vertices black. Hence, $Z_+(K_{4,4} - 3e) = 3$. □

*Proposition* 16. Let $K_{m,n} - 3e$ be a complete bipartite graph with three edges removed. Suppose two edges incident on single vertex and one edge incident on a different vertex are removed. Then

$$Z_+(K_{m,n} - 3e) = \begin{cases} min\{\text{m,n}\} & \text{when } m > n \\ min\{\text{m,n}\} & \text{when } m = n > 3 \\ 2 & \text{when } m = n = 3 \end{cases}$$

*Proof.* Since we are considering only connected graphs we assume $m \geq n \geq 3$.

<u>Case 1</u> Let $m > n$:

Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{m,n}$. Assume that the graph $K_{m,n} - 3e$ is obtained by from $K_{m,n}$ by removing two edges incident on a single vertex and one edge incident on a different vertex. There are two possible cases.

<u>(i)</u> Suppose edges $x_1y_1, x_1y_2, and x_2y_3$ are removed from $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\}, ...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{m,n} - 3e$, we get a star graph with $x_1$ as the center and $y_3, ..., y_n$ as pendants.

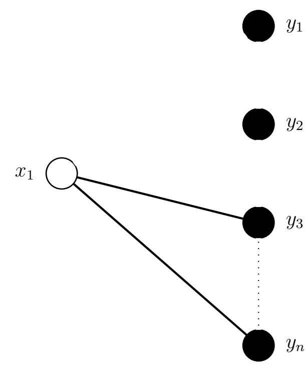



Figure 10.1

So any vertex $y_3, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_2\}$ in $K_{m,n} - 3e$, we get a star graph with $x_2$ as the center and $y_1, y_2, y_4, y_5, y_6, ..., y_n$ as pendants.

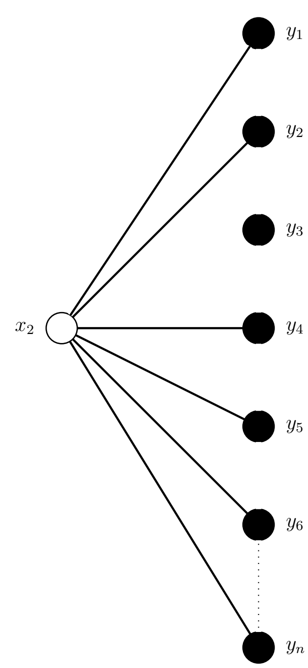



Figure 10.2

So any vertex $y_1, y_2, y_4, y_5, y_6, ..., y_n$ will force $x_2$ black.

If we take the induced graph of $Y \bigcup \{x_3\}$ in $K_{m,n} - 3e$, we get a star graph with $x_3$ as the center and $y_1, y_2, y_4, y_5, y_6, ..., y_n$ as pendants.

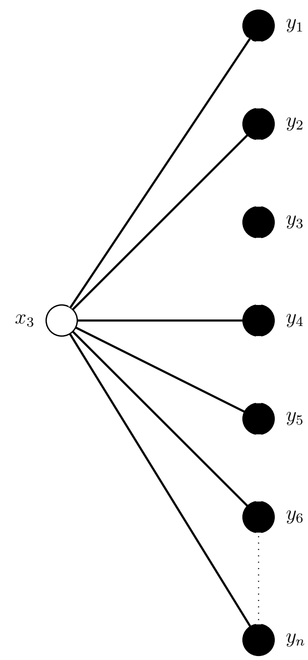



Figure 10.3

So any vertex $y_1, y_2, y_4, y_5, y_6, ..., y_n$ will force $x_3$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - 3e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - 3e) \leq n$.

(ii) Suppose edges $x_1y_1, x_1y_2, and x_2y_1$ are removed from $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\}, ...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{m,n} - 3e$, we get a star graph with $x_1$ as the center and $y_3, ..., y_n$ as pendants.

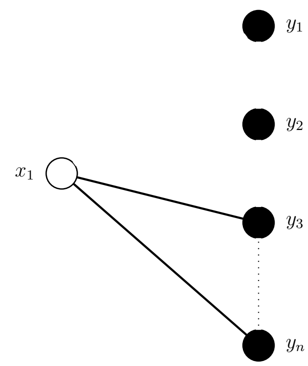



Figure 10.4

So any vertex $y_3, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_2\}$ in $K_{m,n} - 3e$, we get a star graph with $x_2$ as the center and $y_2, ..., y_n$ as pendants.

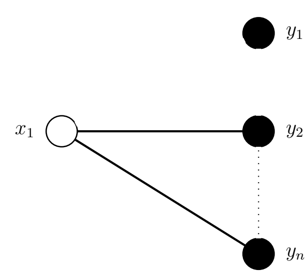



Figure 10.5

So any vertex $y_2, ..., y_n$ will force $x_2$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - 3e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - 3e) \leq n$.

If we color only $n - 1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{m,n} - 3e) = n$.

<u>Case 2</u> Let $m = n > 2$:

There are two subcases.

<u>(i)</u> Suppose edges $x_1y_1, x_1y_2$, and $x_2y_3$ are removed from $K_{n,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{n,n}$, then we get the $m$ components $\{x_1\}, ...\{x_n\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{n,n} - 3e$, we get a star graph with $x_1$ as the center and $y_3, ..., y_n$ as pendants, as in Figure 10.1 So any vertex $y_3, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_2\}$ in $K_{n,n} - 3e$, we get a star graph with $x_2$ as the center and $y_1, y_2, y_4, y_5, y_6, ..., y_n$ as pendants, as in Figure 10.2. So any vertex $y_1, y_2, y_4, y_5, y_6, ..., y_n$ will force $x_2$ black.

If we take the induced graph of $Y \bigcup \{x_3\}$ in $K_{n,n} - 3e$, we get a star graph with $x_3$ as the center and $y_1, y_2, y_4, y_5, y_6, ..., y_n$ as pendants, as in Figure 10.3. So any vertex $y_1, y_2, y_4, y_5, y_6, ..., y_n$ will force $x_3$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq n$ in $K_{n,n} - 3e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq n$. Hence, $Z_+(K_{n,n} - 3e) \leq n$.

<u>(ii)</u> Suppose edges $x_1y_1, x_1y_2$, and $x_2y_1$ are removed from $K_{n,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\}, ...\{x_n\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{n,n} - 3e$, we get a star graph with $x_1$ as the center and $y_3, ..., y_n$ as pendants, as in Figure 10.4. So any vertex $y_3, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_2\}$ in $K_{n,n} - 3e$, we get a star graph with $x_2$ as the center and $y_2, ..., y_n$ as pendants, as in Figure 10.5. So any vertex $y_2, ..., y_n$ will force $x_2$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq n$ in $K_{n,n} - 3e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq n$. Hence, $Z_+(K_{n,n} - 3e) \leq n$.

If we color only $n - 1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{n,n} - 3e) = n$.

Case 3 Let $m = n = 3$:

There are two subcases.

(i) Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be the partite sets of $K_{3,3}$. Assume $K_{3,3} - 3e$ is obtained by removing three edges, $x_1y_1$, $x_1y_2$, and $x_2y_3$, from $K_{3,3}$. If we color $x_2$ and $x_3$ black, then removal of $x_2$ and $x_3$ will give three components namely $\{y_1\}$, $\{y_2\}$, and the edge $x_1y_3$. By the semidefinite color change rule, we can force $y_1$, $y_2$, and $y_3$ black. Again by the semidefinite color change rule $y_3$ will force $x_1$ black. Therefore, $Z_+(K_{3,3} - 3e) \leq 2$. If we color only one vertiex black, then we will get a connected component of white vertices. If we attach the black vertex back to this component we get the induced graph to be $K_{3,3} - 3e$. But the black vertex cannot force any of the white vertices black. Hence, $Z_+(K_{3,3} - 3e) = 2$.

(ii) Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be the partite sets of $K_{3,3}$. Assume $K_{3,3} - 3e$ is obtained by removing three edges, $x_1y_1$, $x_1y_2$, and $x_2y_1$, from $K_{3,3}$. If we color $x_2$ and $x_3$ black, then removal of $x_2$ and $x_3$ will give three components namely $\{y_1\}$, $\{y_2\}$, and the edge $x_1y_3$. By the semidefinite color change rule, we can force $y_1$, $y_2$, and $y_3$ black. Again by the semidefinite color change rule $y_3$ will force $x_1$ black. Therefore, $Z_+(K_{3,3} - 3e) \leq 2$. If we color only one vertex black, then we will get a connected component of white vertices. If we attach the black vertex back to this component we get the induced graph to be $K_{3,3} - 3e$. But the black vertex cannot force any of the white vertices black. Hence, $Z_+(K_{3,3} - 3e) = 2$.

□

*Proposition* 17. Let $K_{m,n} - 3e$ be a complete bipartite graph with three edges removed. Suppose the three edges removed have no common vertices. Then $Z_+(K_{m,n} - 3e) = min\{m, n\}$.

*Proof.* We will assume without loss of generality that $m \geq n$ and show that $Z_+(K_{m,n} - 3e) = n$.

Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{m,n}$. Assume that the graph $K_{m,n} - 3e$ is obtained by removing three edges $x_1$ to $y_1$, $x_2$ to $y_2$, and $x_3$ to $y_3$, from $K_{m,n}$. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the $m$ components $\{x_1\}, ...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{m,n} - 3e$, we get a star graph with $x_1$ as the center and $y_2, ..., y_n$ as pendants.

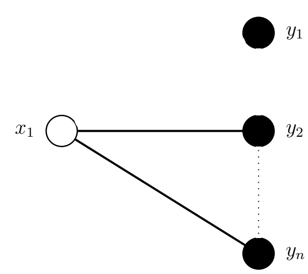



Figure 11.1

So any vertex $y_2, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_2\}$ in $K_{m,n} - 3e$, we get a star graph with $x_2$ as the center and $y_1, y_3, y_4, ..., y_n$ as pendants.

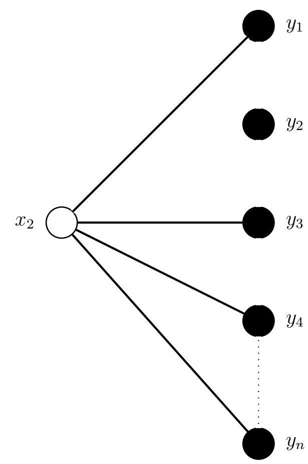



Figure 11.2

So any vertex $y_1, y_3, y_4, ..., y_n$ will force $x_2$ black.

If we take the induced graph of $Y \bigcup \{x_3\}$ in $K_{m,n} - 3$, we get a star graph with $x_3$ as the center and $y_1, y_2, y_4, y_5, y_6, ..., y_n$ as pendants.

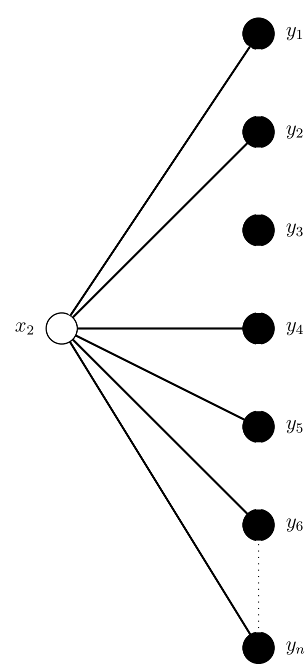



Figure 11.3

So any vertex $y_1, y_2, y_4, y_5, y_6, ..., y_n$ will force $x_3$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - 3e$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - 3e) \leq n$.

If we color only $n - 1$ vertices black, then we will get a connected component of two or more white vertices. If we attach the set of black vertices to this component we get an induced bipartite graph. But the black vertices cannot force all of the white vertices black using the semidefinite color change rule. Hence, $Z_+(K_{m,n} - 3e) = n$. □

*Proposition* 18. Let $K_{m,n} - re$ be a complete bipartite graph with r edges removed, where $m > n > r$. Suppose all the edges removed are incident on a single vertex. Then $Z_+(K_{m,n} - re) = min\{m, n\}$.

*Proof.* Let r be the number of edges removed. We assume that $0 < r < n < m$.

Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$ be the partite sets of $K_{m,n}$. Assume that the graph $K_{m,n} - re$ is obtained from $K_{m.n}$ by removing r edges incident on a single vertex. Suppose we color all the vertices in $Y$ black and color the rest white. By the semidefinite color change rule, if we remove $\{y_1, ..., y_n\}$ from $K_{m,n}$, then we get the m components $\{x_1\}, ...\{x_m\}$ as isolated vertices.

If we take the induced graph of $Y \bigcup \{x_1\}$ in $K_{m,n} - re$, we get a star graph with $x_1$ as the center and $y_{r+1}, ..., y_n$ as pendants.

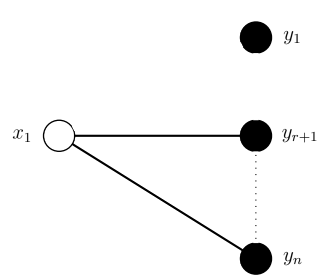



Figure 12.1

So any vertex $y_{r+1}, ..., y_n$ will force $x_1$ black.

If we take the induced graph of $Y \bigcup \{x_k\}$ for $1 < k \leq m$ in $K_{m,n} - re$ we get a star graph with $x_k$ as the center and $y_1, ..., y_n$ as pendants, as before. So any vertex in Y will force $x_k$ black for $1 < k \leq m$. Hence, $Z_+(K_{m,n} - re) \leq n$.

If we color only $n - 1$ vertices black, then we will get a single connected component of white vertices. If we attach the set of black vertices to this component we get the induced graph to be $K_{m,n}$. But none of the black vertices can force any white vertex black. Hence, $Z_+(K_{m,n} - re) = n$. □

*Proposition* 19. Let $K_{m,n} - pe$ be a complete bipartite graph with $p = (m - 1)(n - 1)$ edges removed. Then $Z_+(K_{m,n} - pe) = 1$.

*Proof.* Since $G = K_{m,n} - pe$, where $p = (m - 1)(n - 1)$, is a spanning tree of $K_{m,n}$, we color any vertex of $G$ black. Since $G$ is a tree the result follows from [2]. □

## References


[1] AIM Minimum Rank - Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioaba, D. Cvetkovic, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanovic, H. van der Holst, K. Vander Meulen, and A. Wangsness), Zero Forcing sets and the minimum rank of graphs, *Linear Algebra and its Applications*, **428/7**, (2009), 1628-1648.

[2] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, Zero forcing paremeters and minimum rank problems, *Linear Algebra and its Applications*, **433**(2010) 401-411.

[3] F. Barioli, S. Fallat, L. Mitchell, and S. Narayan. Minimum semidefinite rank of outerplanar graphs and the tree cover number, *Electronics Journal of Linear Algebra*, **22** (2011) 10-21.

[4] J. Beagley, L. Mitchell, S. Narayan, E. Radzwion, S. Rimer, R. Tomasino, J. Wolfe, and A. Zimmer, Bounds for minimum rank problems from superpositions and cut sets, *Linear Algebra and its Applications*, **438** (2013) 4041-4061.

[5] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press Ltd, 264, 1976.

[6] M. Booth, P. Hackney, B. Harris, C. Johnson, M. Lay, T. Lenker, L. Mitchell, S. Narayan, A. Pascoe, and B.D. Sutton, On the minimum semidefinite rank of a simple graph, *Linear and Multilinear Algebra*, **59** (2011) 483-506.

[7] M. Booth, P. Hackney, B. Harris, C. Johnson, M. Lay, L. Mitchell, S. Narayan, A. Peascoe, K. Steinmetz, B. Sutton, and W. Wang, On the minimum rank among positive semidefinite

matrices with a given graph, *SIAM Journal on Matrix Analysis and Applications*, **30-2** (2008) 731-740.

[8] R. Cranfill, L. Mitchell, S. Narayan and T. Tsutsui, Maximum induced trees and minimum semidefinite rank, Involve 5 (2012) 411-420.

[9] S. Fallat and L. Hogben, The minimum rank of symmetric matrices described by a graph: a survey, *Linear Algebra and its Applications*, **426** (2007) 558-582.

[10] P. Hackney, B. Harris, M. Lay, L. Mitchell, S. Narayan, and A. Pascoe, Linearly independent vertices and minimum semidefinite rank, *Linear Algebra and its Applications*, **431** (2009) 1105-1115.

[11] R.A. Horn, C.R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, Cambridge, 2013.

[12] T. Jensen, B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.

[13] L. H. Mitchell, S. K. Narayan, and A. M. Zimmer, Lower bounds in minimum rank problems, *Linear Alebra and its Applications*, **432** (2010) 430-440.

[14] L. H. Mitchell, S. K. Narayan, and A. M. Zimmer, Lower bounds for minimum semidefinite rank from orthogonal removal and chordal supergraphs, *Linear Algebra and its Applications*, **436** (2012) 525-536.

[15] H. van der Holst, Graphs whose positive semidefinite matrices have nullity at most two, *Linear Algebra and its Applications*, **375** (2003) 1-11.

[16] H. van der Holst, On the maximum positive semidefinite nullity and the cycle matroid of graphs, *Electronic Journal of Linear Algebra*, **18** (2009) 192-201.

[17] D. B. West, *Introduction to Graph Theory*, 2nd ed. Prentice-Hall, Inc., Upper Saddle River, NJ, 2001.