TREE COVER NUMBER AND MAXIMUM SEMIDEFINITE NULLITY OF GRAPHS

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A thesis submitted in partial fulfillment of
the requirements for the degree of
Master of Arts

Department of Mathematics

Central Michigan University
Mount Pleasant, Michigan
February 2017

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. Sivaram K. Narayan, for all of his help, support, and friendship during my undergraduate and graduate years. Without his encouragement and guidance, this work would not be possible. I would also like to thank my committee members, Dr. Jim Angelos and Dr. Ben Salisbury for their assistance and support in this project.

In addition, I would like to thank my family, partner, and friends for their endless love and support.

Finally, I would like to thank all of the faculty and staff members of the mathematics department at Central Michigan University for creating such a supportive and wonderful home during my time here.

ABSTRACT

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by Rachel Jade Domagalski

A graph G consists of a set of vertices $\{v_1,\ldots,v_n\}$ and a set of edges which are unordered pairs of vertices. A multigraph consists of possibly parallel edges but not loops. For a given multigraph G, we can associate a complex Hermitian matrix $A=[a_{ij}]$ as follows: $a_{ij}=0$ if $i\neq j$ and v_i,v_j are nonadjacent, $a_{ij}\neq 0$ if $i\neq j$ and v_i,v_j are ends of a single edge, a_{ij} is a complex number if $i=j$ or if $v_i,v_j,i\neq j$ are joined by parallel edges. In this work, we consider the collection of positive semidefinite complex Hermitian matrices associated with a given multigraph G, denoted $S_+(G)$. The minimum of rank A for all matrices A in $S_+(G)$ is called the minimum semidefinite rank of G, denoted $mr_+(G)$. The corresponding maximum semidefinite nullity, denoted by $M_+(G)$, satisfies $mr_+(G)+M_+(G)=|G|$, where $|G|$ is the number of vertices in G.

In order to compute $M_+(G)$ for a given graph G, certain graph parameters have been developed. One such graph parameter is called the tree cover number of G, denoted $T(G)$. This is the minimum number of vertex disjoint simple trees occurring as induced subgraphs of G that cover all vertices of G. This parameter is easier to compute than $M_+(G)$. Also, it has been conjectured that, for any multigraph G, $T(G)\leq M_+(G)$.

In this thesis, we find the tree cover number of line graphs, shadow graphs, corona of two graphs, cartesian product of two graphs, and a few more. In these cases, we verify that the above conjecture holds.

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CHAPTER I

INTRODUCTION

In this thesis, we find the tree cover number $T(G)$ of certain classes of graphs. In particular, we examine the relationship between the tree cover number of two graphs G_1 and G_2 and the new graph G generated by using graph operations such as the cartesian product, lexicographic product, and corona. We also examine the connection between tree cover number of G and that of its shadow graph $S(G)$ and line graph $L(G)$. In addition to finding these relationships, we verify whether or not the conjecture $T(G) \leq M_+(G)$ [3] holds for each new graph under the above mentioned graph operations, where $M_+(G)$ is the maximum semidefinite nullity of G.

Maximum nullity and minimum rank problems have connections to other combinatorial matrix theory problems, such as the Inverse Eigenvalue Problem of a graph. The Inverse Eigenvalue Problem asks to determine, for a given graph G, what eigenvalues are possible for a real symmetric matrix A having nonzero off-diagonal entries determined by the adjacency of G [12]. Thus, it is of interest to gain a better understanding of the connections between the maximum semidefinite nullity and graph parameters, such as the tree cover number.

In Chapters 2 and 3, we present preliminary graph and matrix theory information as well as define the tree cover number $T(G)$ and maximum semidefinite nullity $M_+(G)$ of a graph G. We also discuss the conjecture, proposed in [3], that $T(G) \leq M_+(G)$. Here, we also present known results that we use in our comparisons of tree cover number to the maximum semidefinite nullity. In Chapter 4, we discuss minimum semidefinite rank (*msr*) and methods that are used to find the $msr(G)$ for a given graph G. Our goal in this research is two-fold: first, to find the tree cover number of different classes of graphs using different graph operations, and then to verify whether or not this conjecture holds.

We prove in Chapter 5 that the tree cover number of the line graph $L(G)$ is the minimum number of paths needed in a path decomposition of a simple connected graph G. Finding the path decomposition number is easier than computing the tree cover number for a line graph. We also show that the tree cover number $T(L(G))$ of the line graph $L(G)$ is bounded below by the tree cover number $T(G)$ of the graph G. Examples of graph classes are presented in which equality holds and where there is strict inequality.

In Chapter 6, we investigate the tree cover number of a shadow graph. The shadow graphs of paths, cycles, complete graphs, and complete bipartite graphs are considered. For the shadow graph and p-shadow graph of a path, we show the tree cover number is equal to the maximum semidefinite nullity.

We consider the tree cover number of the corona of two graphs $G \circ H$ in Chapter 7. An upper bound on the tree cover number based on $T(G)$ and $T(H)$ is given. We provide a class of graphs for which this bound is sharp. In addition, we prove that the conjecture, $T(G) \leq M_+(G)$, holds for the corona graph when each of the two graphs individually satisfy the conjecture.

In Chapter 8, two types of graph products are considered; namely, the cartesian product and lexicographic product. We find the tree cover number for different types of cartesian products and verify that the maximum semidefinite nullity conjecture holds. We discuss the lexicographic product of two paths. This result shows that, for the two different graph products for paths, the tree cover numbers are quite different.

Additional results on the tree cover number are provided in Chapter 9. These results include finding the tree cover number for complete m-partite graphs, wheel graphs, and gear graphs. We verify the conjecture for these cases. We also consider the effect of vertex deletion on the tree cover number of a graph G. We find a lower bound for $T(G-v)$ and give a necessary and sufficient condition for when this bound is attained. A sufficient condition for when the vertex deletion has no effect on the tree cover number of G is provided.

CHAPTER II

PRELIMINARIES

II.1. Graph Theory

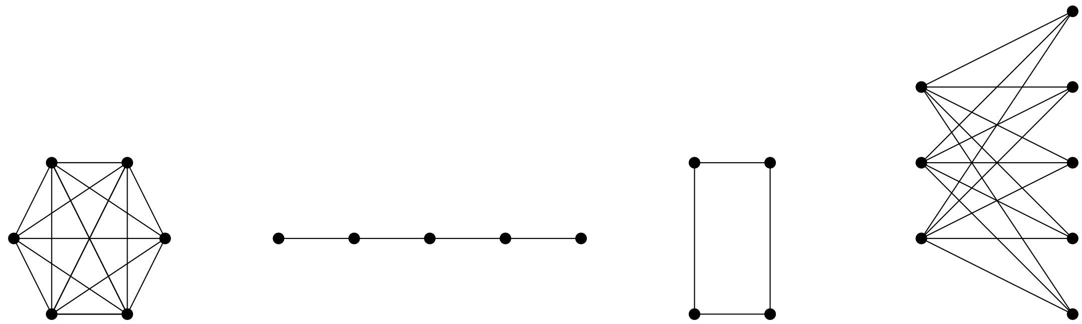
A *simple graph* G consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G)$ consisting of unordered pairs of vertices. The number of vertices in G is called the *order* of G and denoted by $|G|$. Write $e = uv$ for an edge e with endpoints u and v. If u and v are the ends of an edge, they are said to be *adjacent* or *neighbors*. The ends of an edge are said the be *incident* with the edge, and vice-versa. The set of neighbors of a vertex v in G is denoted $N_G(v)$. The number of edges of G incident with v is called the *degree* of v in G, denoted by $d_G(v)$. For a simple graph, $d_G(v) = |N_G(v)|$. A vertex of degree one is called a pendant vertex or a leaf of G.

An *independent set* in a graph is a set of pairwise nonadjacent vertices. The maximum cardinality among independent sets in G is called the *independence number*, denoted by $\alpha(G)$. A *clique* of a graph G is a set of mutually adjacent vertices.

A *complete graph* K_n on n vertices is a simple graph where any two vertices are adjacent. A graph is said to be *bipartite* if its vertex set can be partitioned into two subsets X and Y, called *parts*, so that every edge has one end in X and one end in Y. If every vertex in X is joined to every vertex in Y, then G is called a complete bipartite graph, denoted $K_{m,n}$ where $|X| = m$ and $|Y| = n$. A similar definition can be given for a complete multipartite graph.

A *path* is a simple graph whose vertices can be arranged in a linear sequence $v_1 v_2 \ldots v_n$ such that two vertices are adjacent if and only if they are consecutive in the sequence. The path on n vertices is denoted P_n. A *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence so that two vertices are adjacent if they are consecutive in the sequence and nonadjacent otherwise. The cycle on n vertices is denoted by C_n. The length of P_n or C_n is the number of edges in P_n or C_n. Examples of these graphs can be seen in Figure 1.

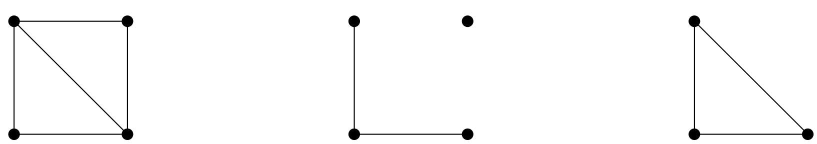
A graph with no cycle is called acyclic. A tree is a connected acyclic graph. One type of tree is called a *star*, denoted St_n, which is the graph $K_{1,n-1}$.



K_6 P_5 C_4 $K_{3,5}$

Figure 1. Above are several standard graphs: K_6, P_5, C_4, and $K_{3,5}$.

A graph G is *connected* if there exists a path in G between any two vertices of G. Given a graph G, F is a *subgraph* of G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. An *induced subgraph* of G, denoted by $G[Y]$, is the subgraph of G whose vertex set is Y and whose edge set consists of all edges in $E(G)$ which have both ends in Y. An example of a subgraph and an induced subgraph can be seen in Figure 2.



G H F

Figure 2. The graph H is a subgraph of G. The graph F is an induced subgraph of G.

A graph G is said to be *chordal* if it has no induced cycles of length four or more. A graph which can be drawn in the plane in such a way that the edges meet only at points corresponding to their common ends is called a *planar graph*, and such a drawing is called a *planar embedding* of the graph. A graph is *outerplanar* if it has a crossing-free planar embedding such that all vertices are on the same face. Equivalently, an outerplanar graph has no subgraph homeomorphic to K_4 or $K_{2,3}$. For further details in graph theory, one may consult [7] or [22].

II.2. Matrix Theory

The set of $n \times n$ matrices with entries that are complex numbers is denoted by $M_n(\mathbb{C})$ and those with entries that are real numbers is denoted by $M_n(\mathbb{R})$. A matrix $A \in M_n(\mathbb{C})$ is said to be Hermitian if A equals its conjugate transpose A^*. A matrix $A \in M_n(\mathbb{R})$ is said to be symmetric if A equals its conjugate.

If $A \in M_n(\mathbb{C})$ and α, β are subsets of $\{1, 2, \ldots, n\}$, we denote by $A[\alpha, \beta]$ the submatrix of A whose rows are indexed by α and whose columns are indexed by β. We write $A[\alpha, \alpha]$ as $A[\alpha]$ and call it a principal submatrix of A.

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ are points in $\mathbb{C}^n$, then the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$. A Hermitian matrix $A \in M_n(\mathbb{C})$ is *positive definite* if $\langle A\vec{x}, \vec{x} \rangle$ is positive for all nonzero $\vec{x}$ in $\mathbb{C}^n$ and it is *positive semidefinite* if $\langle A\vec{x}, \vec{x} \rangle \geq 0$ for all $\vec{x}$ in $\mathbb{C}^n$. If A is positive semidefinite then all its principal submatrices are positive semidefinite. A Hermitian matrix is positive semidefinite if and only if all of its eigenvalues are nonnegative.

Theorem 1 ([15, p. 440])**.** *A matrix $A \in M_n(\mathbb{C})$ is positive semidefinite if and only if there exists $B \in M_{m,n}(\mathbb{C})$ such that $A = B^*B$.*

Theorem 2 ([15, p. 441])**.** *Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in $\mathbb{C}^m$ and let $G = [\langle \vec{v}_i, \vec{v}_j \rangle]_{i,j=1}^{n} \in M_n$. Then*

(a) G is Hermitian and positive semidefinite

(b) $\operatorname{rank}(G) = \dim \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$.

In the above theorem, G is called the Gram matrix of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. From the last two theorems, we conclude that every positive semidefinite matrix is a Gram matrix and every Gram matrix of vectors is positive semidefinite.

We may think of $A \in M_{m,n}(\mathbb{C})$ as a linear transformation $\vec{x} \mapsto A\vec{x}$ from $\mathbb{C}^n$ to $\mathbb{C}^m$. We define for $A \in M_{m,n}(\mathbb{C})$, the nullspace of A and the range of A as follows: nullspace $A = \{\vec{x} \in \mathbb{C}^n : A\vec{x} = 0\}$

and range $A = \{\vec{y} \in \mathbb{C}^m : \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{C}^n\}$. The dimension of the nullspace A is denoted by nullity$A$ and the dimension of the range A is denoted by rank$A$. These numbers are related by the *rank-nullity theorem*:

$$\dim(\text{range } A) + \dim(\text{nullspace } A) = \text{rank} A + \text{nullity} A = n.$$

For further results in Matrix Theory, one may consult [15].

CHAPTER III

TREE COVER NUMBER AND MAXIMUM SEMIDEFINITE NULLITY

Let $G = (V,E)$ denote an undirected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ which has no loops but possibly has parallel edges (a multigraph). Given a multigraph G, we associate a real symmetric or a complex Hermitian matrix $A = [a_{ij}]$ in the following manner:

- $a_{ij} = 0$ if $i \neq j$ and v_i, v_j are nonadjacent,
- $a_{ij} \neq 0$ if $i \neq j$ and v_i, v_j are joined by a single edge, and
- a_{ij} is unrestricted if $i = j$ or v_i, v_j are joined by parallel edges.

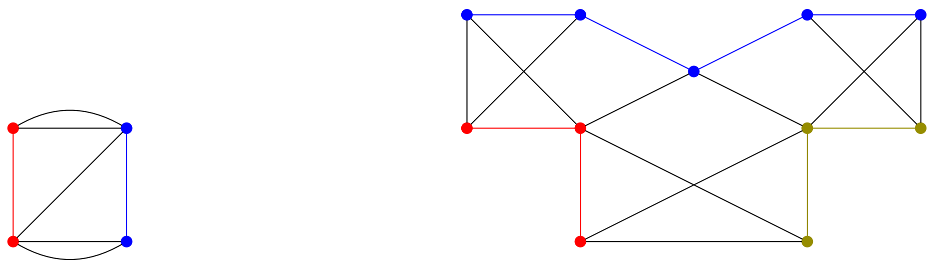
Let $S_+(G,\mathbb{R})$ (respectively $S_+(G,\mathbb{C})$) denote the subsets of positive semidefinite (psd) real symmetric (respectively complex Hermitian) matrices whose graph is G. Note that if $A \in S_+(G,\mathbb{F})$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, the diagonal entry $a_{ii} \geq 0$ (since principal submatrices of psd matrices are psd). Moreover, if there is a nonzero off-diagonal entry in the ith row of A, then $a_{ii} > 0$.

By $M_+(G,\mathbb{F})$, we denote the maximum possible nullity of any matrix $A \in S_+(G,\mathbb{F})$. The smallest possible rank of any matrix $A \in S_+(G,\mathbb{F})$ is denoted $mr_+(G,\mathbb{F})$ and is called the *minimum semidefinite rank* of G. From the above definitions, it follows that $M_+(G,\mathbb{F}) + mr_+(G,\mathbb{F}) = |G|$ where $|G|$ is the number of vertices in G.

Since $S_+(G,\mathbb{R}) \subseteq S_+(G,\mathbb{C})$, we conclude that $mr_+(G,\mathbb{C}) \leq mr_+(G,\mathbb{R})$. An example where strict inequality holds is given in [2]. When $mr_+(G,\mathbb{C}) = mr_+(G,\mathbb{R})$ we omit the field reference in the notation and write $msr(G)$ for the minimum semidefinite rank of G.

In general, it is difficult to find $msr(G)$ for a given graph G. Therefore, one method is to gain a better understanding between $msr(G)$ and known graph parameters. In [9], it was observed that the independence number $\alpha(G)$ is a lower bound for $msr(G)$. It has been shown that $msr(G)$ is exactly the *clique cover number* of G whenever G is a chordal graph. That is, $msr(G)$ for a chordal graph G equals the minimum number of cliques needed to cover all the (single) edges of a multigraph G (see [9, 13]).

Another method is to define new graph parameters and study how they are related to either $M_+(G)$ or $msr(G)$. One such parameter, called the *tree cover number* of G and denoted $T(G)$, was defined and studied in [3]; specifically, it is the minimum number of vertex disjoint simple trees occurring as induced subgraphs of G that cover all the vertices of G.



$T(G) = 2$ $\qquad$ $T(H) = 3$

Figure 3. Here are a couple of examples giving the trees in a minimal tree cover. The trees are shown in red, blue, and olive.

It is clear that the tree cover number of a simple tree is one and the tree cover number of a simple unicyclic graph (graphs with exactly one cycle) is two. It is easy to observe that $T(K_n) = \lceil \frac{n}{2} \rceil$. Indeed, if n is even, the edges $\{v_1v_2, v_3v_4, \ldots, v_{n-1}v_n\}$ form a tree cover of K_n. If n is odd, the tree cover consists of $\frac{n-1}{2}$ edges and a single vertex, namely $\{v_1v_2, v_3v_4, \ldots, v_{n-2}v_{n-1}\} \cup \{v_n\}$. If $G = K_{m,n}$, one can show $T(K_{m,n}) = 2$. In this case, we take the two stars induced by the last vertex of one partite set with all but the last vertex of the other partite set.

We now recall some results from [3]. It was shown that if G is a multigraph that is outerplanar, then $T(G) = M_+(G)$. The result was extended to partial 2-trees with essentially the same proof in [10]. Moreover, it was conjectured that $M_+(G) \geq T(G)$, for any multigraph G. It has been shown that $M_+(G) \geq T(G)$ for any chordal multigraph and certain simple bipartite graphs.

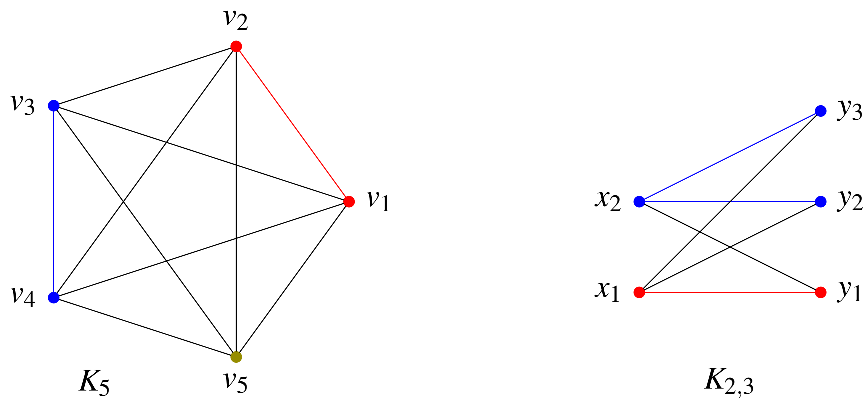


Figure 4. Above are the graphs of K_5 and $K_{2,3}$. The trees of a minimal tree cover are shown in red, blue, and olive.

Also, if G and H satisfy the conjecture then so does their join $G \vee H$ and their vertex sum $G \cdot H$ [3].

In this research, our goal is two-fold: one is to find the tree cover number of different classes of graphs, and the other is to verify if the conjecture $M_+(G) \geq T(G)$ holds for these graphs.

CHAPTER IV

FINDING MINIMUM SEMIDEFINITE RANK

In this chapter, we will state some known results about minimum semidefinite rank that will be used in the subsequent chapters. This chapter could be skimmed through in the first reading and revisited when needed. Henceforth, we will abbreviate positive semidefinite as *psd* and minimum semidefinite rank as *msr*.

IV.1. Lower Bounds

Lemma 3 ([9]). *If F is an induced subgraph of a simple connected graph G, then*

$$msr(F) \leq msr(G).$$

For a graph G, we define its *tree size*, denoted $ts(G)$, to be the number of vertices in a maximum induced tree [11]. It is known that $msr(G) = |G| - 1$ if and only if G is a tree [20]. This fact combined with Lemma 3 gives the following result.

Proposition 4 ([9]). *If G is a simple connected graph, then $msr(G) \geq ts(G) - 1$.*

As a corollary we get *msr* of cycles.

Corollary 5 ([9]). *If G is a cycle on n vertices, then $msr(G) = |G| - 2$.*

For an induced forest of G with components $T_1, T_2, \ldots, T_k$, count

$$fm(G) := ts(T_1) + ts(T_2) + \cdots + ts(T_k) - (\text{the number of isolated vertices in the forest}).$$

This number $fm(G)$ is called the *forest measure* of G.

Proposition 6 ([9]). *If G is a simple connected graph, then $msr(G) \geq fm(G) \geq ts(G) - 1$.*

Corollary 7 ([9]). *For a simple connected graph G, $msr(G) \geq \alpha(G)$ where $\alpha(G)$ is the independence number of G.*

IV.2. Vertex Sum

It is known that $msr(G) = 1$ for a simple connected graph G on two or more vertices if and only if G is a complete graph on $|G|$ vertices. In this section, we discuss the *msr* of a vertex sum.

Definition 1. *If G is a graph with a cut vertex v, such that H_1 and H_2 are the connected components of the vertex deletion $G - v$, then we write $G = G_1 \cdot G_2$ where G_1 and G_2 are the subgraphs of G induced by $V(H_1) \cup \{v\}$ and $V(H_2) \cup \{v\}$, respectively.*

Theorem 8 ([8])**.** *If $G = G_1 \cdot G_2$, then $msr(G) = msr(G_1) + msr(G_2)$.*

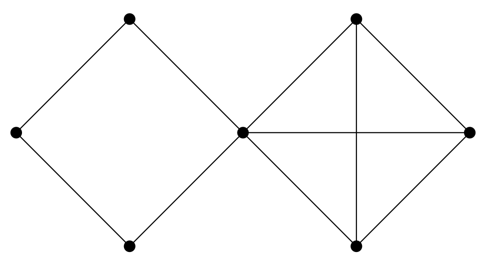


Figure 5. The graph above is a vertex sum of C_4 and K_4. Therefore, the minimum semidefinite rank of this graph is $msr(C_4) + msr(K_4) = 2 + 1 = 3$.

IV.3. Vector Representations

Let $\vec{X} = \{\vec{x}_1, \ldots, \vec{x}_n\}$ be a set of n nonzero vectors in $\mathbb{C}^m$. Let $X = [\vec{x}_1 \ldots \vec{x}_n]$ be the $m \times n$ matrix with $\vec{x}_1, \ldots, \vec{x}_n$ as the column vectors. Then X^*X is a psd matrix called the Gram matrix of $\vec{X}$ with regard to the Euclidean inner product. Its associated simple graph G has n vertices $v_1, \ldots, v_n$ which correspond to the vectors $\vec{x}_1, \ldots, \vec{x}_n$ and single edges corresponding to the nonzero inner products among those vectors. By rank$(\vec{X})$, we mean the dimension of span$\{\vec{x}_1, \ldots, \vec{x}_n\}$, which is equal to the rank of X^*X [15]. We say $\vec{X} = \{\vec{x}_1, \ldots, \vec{x}_n\} \subset \mathbb{C}^m$ is a *vector representation* of G when X^*X is a complex Hermitian psd matrix corresponding to G [17]. From Theorem 1 and Theorem 2 in Chapter 2, we see that finding a psd matrix with graph G and finding a vector representation of

G are equivalent problems. Therefore, the smallest m for which there exists a vector representation of G in $\mathbb{C}^m$ is equal to $msr(G)$.

Figure 6. The above graphs G and H have a vector representation in $\mathbb{C}^2$. Since G and H are not complete graphs, $msr(G) = 2 = msr(H)$.

IV.4. Orthogonal Removal of Vertices

Let $\vec{V}$ be a vector representation of a graph G and v be a fixed vertex of G with vector $\vec{v} \in \vec{V}$. We orthogonally project each vector $\vec{v}_j \in \vec{V}$ onto the complement of the span of $\vec{v}$. That is, replace each $\vec{v}_j$ in $\vec{V}$ with $\vec{v}_j - \frac{\langle \vec{v}, \vec{v}_j \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$ to get a vector representation $\vec{V} \ominus \vec{v}$ with rank one less than $\vec{V}$ of a graph G' with order one less than G. To reflect the above situation we now define a multigraph $G \ominus v$ such that $\vec{V} \ominus \vec{v}$ is a vector representation of $G \ominus v$.

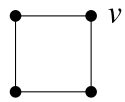
Consider the induced subgraph $G - v$ of G. Now add p edges between each pair $u, w \in N(v)$, where p is the product of the number of edges between u and v and the number of edges between w and v to obtain the supergraph $G \ominus v$ of $G - v$.

We say that a vertex v in a multigraph G is *singly-isolated* if the set of vertices of G adjacent to v by exactly one edge is empty.

Proposition 9 ([9]). *For any multigraph G with a non-singly isolated vertex v,*

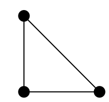
$$msr(G \ominus v) \leq msr(G) - 1.$$

The following theorems give conditions under which equality holds in the inequality of the above proposition. A vertex is said to be *simplicial* in G if $N(v)$ induces a complete graph.



$$\vec{V} = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\}$$

$$\vec{V} \ominus \vec{v} = \left\{ \begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$$

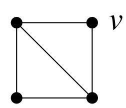


G $\quad$ $G \ominus v$

Figure 7. Above is the graph G and the graph $G \ominus v$ which has vector representation $\vec{V} \ominus \vec{v}$.

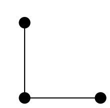
Theorem 10 ([3, 9])**.** *Suppose v is a vertex of G that is not singly isolated. If v is either simplicial or $d_G(v) \le 2$, then $msr(G) = msr(G \ominus v) + 1$ (or equivalently, $M_+(G) = M_+(G \ominus v)$).*

Theorem 11 ([13])**.** *Let G be a connected graph, let v be a vertex of G not adjacent to any of its neighbors by multiple edges, and let H be the graph induced by the vertices of $N(v)$. If the complement of H is a star forest then $msr(G) = msr(G \ominus v) + 1$.*

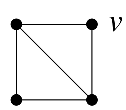


$$\vec{V} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

$$\vec{V} \ominus \vec{v} = \left\{ \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix} \right\}$$

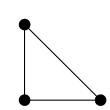


G $\quad$ Graph for $\vec{V} \ominus \vec{v}$

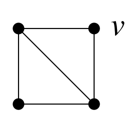


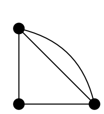
$$\vec{V} = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$$

$$\vec{V} \ominus \vec{v} = \left\{ \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$$



G $\quad$ Graph for $\vec{V} \ominus \vec{v}$





G $\quad$ $G \ominus v$

Figure 8. For the above graph G, different vector representations result in different graphs for $\vec{V} \ominus \vec{v}$. We depict $G \ominus v$ with a multiple edge to represent that these different cases may occur.

CHAPTER V

TREE COVER NUMBER OF LINE GRAPHS

In this chapter, we prove that the tree cover number of the line graph $L(G)$ is the minimum number of paths needed in a path decomposition of a simple connected graph G. In addition, we prove that the tree cover number of the line graph $L(G)$ is bounded below by the tree cover number of the graph G. We provide cases where equality is obtained and compare these results to the maximum semidefinite nullity of the graph G.

V.1. Tree Cover Number of Line Graphs

Let G be a simple connected graph with n vertices and m edges. The *line graph* of a graph G, denoted $L(G)$, has vertex set consisting of the m edges of G and an edge between two vertices if and only if the corresponding edges in G have a common end vertex. For example, the kite $L(G)$ is the line graph of the paw graph G, as seen in Figure 9.

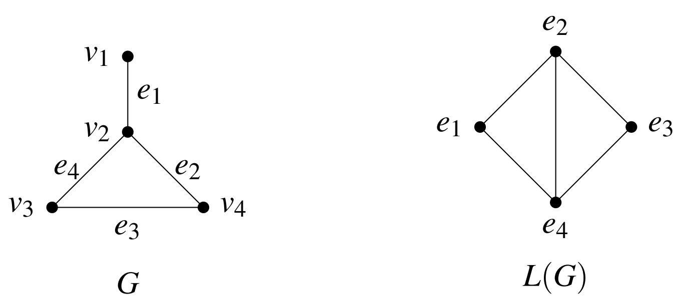


Figure 9. The graph G is the paw graph. It consists of four edges, e_1, e_2, e_3, e_4. These four edges correspond to the four vertices in $L(G)$, the kite graph. The vertex e_2 in $L(G)$ is adjacent to e_1, e_3, e_4 because edge e_2 in G is adjacent to the edges e_1, e_3, e_4 in G.

In order to find the tree cover number of $L(G)$, we first show that every induced simple tree in $L(G)$ must be a path. This result is a consequence of the following theorem. A claw in a graph G is defined to be an induced subgraph isomorphic to $K_{1,3}$. A triangle graph S is the graph

K_3. A triangle S in G is odd if $|N(v) \cap V(S)|$ is odd for some $v \in V(G)$. A triangle S in G is even if $|N(v) \cap V(S)|$ is even for every $v \in V(G)$ [22].

Theorem 12 ([5, 6, 16, 21])**.** *Let H be a graph. The following conditions are equivalent:*

1. *H is a line graph of G; that is, $L(G) = H$;*

2. *The edges of H can be partitioned into complete subgraphs such that no vertex lies in more than two of the subgraphs;*

3. *H is claw-free and if two odd triangles have a common edge, then the subgraph induced by their vertices is a K_4;*

4. *H contains none of the nine forbidden graphs of Figure 10 as an induced subgraph.*

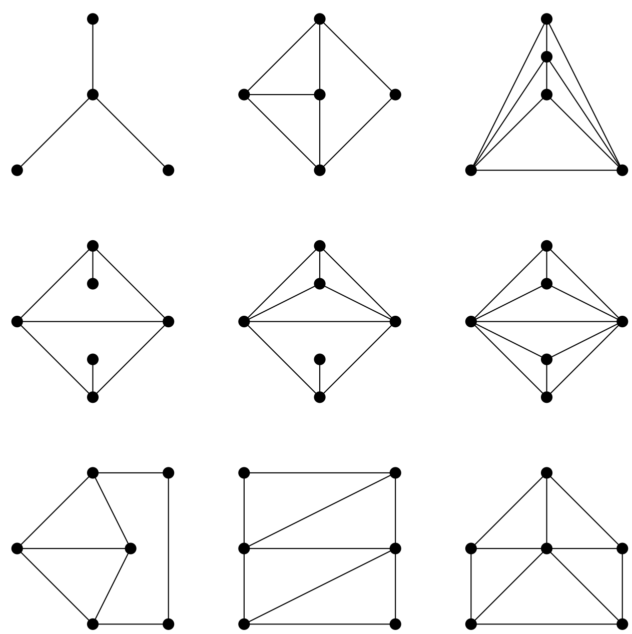


Figure 10. These nine graphs are forbidden induced subgraphs of any line graph.

Lemma 13. *Let $L(G)$ be the line graph of a simple graph G. Then every induced simple tree in $L(G)$ must be an induced path in $L(G)$.*

Proof. Suppose T is an induced simple tree in $L(G)$ and v is a vertex of T. If $\deg(v) \geq 3$ in $L(G)$, then v and three of its neighbors will induce a $K_{1,3}$ in $L(G)$, contradicting the characterization given in Theorem 12. Therefore, every vertex v of T has degree at most two. Thus T must be an induced path in $L(G)$. □

Proposition 14. *Any path in G corresponds to an induced path in $L(G)$. Moreover, every induced path in $L(G)$ is obtained from a path in G.*

Before we present the proof, note that in Figure 9, $P = v_1v_2v_3v_4$ is not an induced path in G, but the corresponding path $Q = e_1e_4e_3$ is an induced path in $L(G)$.

Proof. Suppose $P = v_1e_1v_2e_2\dots v_le_lv_{l+1}$ is a path in G. Then clearly $Q = e_1e_2\dots e_{l-1}e_l$ is a path in $L(G)$ since adjacent edges in P have a common vertex. In order to show that Q is an induced path in $L(G)$, we show that the vertices $e_1e_2\dots e_l$ do not induce a cycle amongst themselves. Suppose a cycle is induced, so that $\deg(e_j) \geq 3$, for some j such that $1 < j < l$. Then a $K_{1,3}$ is induced in $L(G)$, contradicting the fact that $L(G)$ is a line graph (See Theorem 12). If e_l and e_1 are adjacent in $L(G)$ then $v_{l+1} = v_1$ in P, which contradicts the assertion that the vertices of a path in G are distinct. Therefore, Q is an induced path in $L(G)$.

Suppose $e_1e_2\dots e_{l-1}e_l$ is an induced path in $L(G)$. Since the vertices are distinct in $L(G)$, the edges in G are distinct and the adjacent edges have a common vertex. Thus, $P = v_1e_1v_2e_2\dots v_le_lv_{l+1}$ is a path in G. □

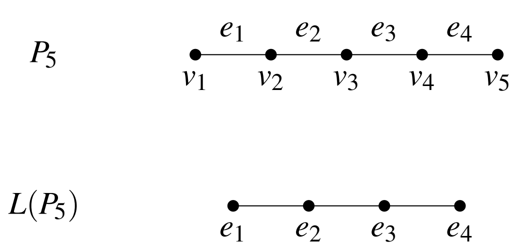


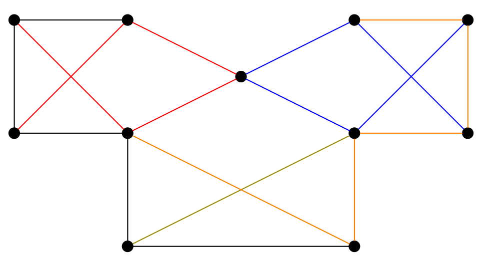
Figure 11. The line graph of a simple path on 5 vertices, $L(P_5)$, is a path on 4 vertices. Thus a P_5 in a graph G corresponds to an induced P_4 in $L(G)$.

From Lemma 13 and Proposition 14, it is clear that knowing the path decomposition of G is useful in finding tree covers of $L(G)$.

Definition 2. *A* path decomposition *of a graph G is a list of paths* $\{P_1, P_2, \ldots, P_k\}$ *such that each edge of G appears in exactly one path in the list. The minimum number of paths in such a list is called the path decomposition number of G and is denoted by* $p(G)$.

Theorem 15. *Let* $L(G)$ *be the line graph of a simple graph G. Then* $T(L(G)) = p(G)$.

Proof. Let k be the number of paths in a minimal path decomposition of G. Each of the paths $P_1, \ldots, P_k$ in the minimal path decomposition produces induced paths $Q_1, \ldots, Q_k$ in $L(G)$. Since each edge in G appears in one and only one path in $\{P_1, \ldots, P_k\}$, we conclude that each vertex of $L(G)$ appears in one and only one path in $\{Q_1, \ldots, Q_k\}$. Thus we have a vertex disjoint tree cover of $L(G)$ of size k. Hence $T(L(G)) \leq k$. By Lemma 13, every induced tree in $L(G)$ has to be an induced path in $L(G)$. From Proposition 14, every induced path in $L(G)$ is obtained from a path in G. Therefore a minimal tree cover in $L(G)$ must be obtained from a path decomposition in G. Hence $k \leq T(L(G))$. Thus $T(L(G)) = p(G)$. □



G

Figure 12. In the above graph G, $p(G) = 5$, shown above in five different colors. Finding the path decomposition is easier than finding the tree cover number of $L(G)$ which has order 19.

Theorem 16. *Let* $L(G)$ *be the line graph of a simple graph G. Then* $T(G) \leq T(L(G))$.

Proof. From Theorem 15 it is enough to show that $T(G) \leq p(G)$. Let $\{T_1, \ldots, T_k\}$ be a minimal tree cover of G. If one of the trees T_i, for some $1 \leq i \leq k$, is not a path, then more than one path is needed to cover the edges of T_i. Hence $k \leq p(G)$. Now suppose all the trees are induced simple paths in G. If all the edges of G are covered by the k paths, then $k = p(G)$. If some of the edges of G are not covered by the k paths then vertices of those edges must induce a cycle with vertices of some path in the cover. Suppose e and f are two edges not covered by the paths $T_1, \dots, T_k$. In addition, suppose e and f induce a cycle with some vertices of T_1. Then at least one additional path is needed to cover edges e and f. Hence $k \leq p(G)$. □

Equality in the above theorem is possible as seen from the following proposition.

Theorem 17 ([14])**.** *Let K_n be the complete graph on n vertices, then $p(K_n) = \lceil \frac{n}{2} \rceil$.*

Proposition 18. *If G is either a simple path, a cycle, or a complete graph, then $T(G) = T(L(G))$.*

Proof. We first consider the case $G = P_m$, where P_m is a path on m vertices. Let $V(P_m) = \{v_1, \dots, v_m\}$ with edges $e_i = v_i v_{i+1}$ for $i = 1, \dots, m-1$. Then $p(P_m) = 1$, as the path cover is exactly P_m. Hence $T(L(P_m)) = 1$ by Theorem 15.

Suppose the graph $G = C_m$, where C_m is a cycle on m vertices. Let $V(C_m) = \{v_1, \dots, v_m\}$ with edges $e_i = v_i v_{i+1}$ for $i = 1, \dots, m-1$ and $e_m = v_1 v_m$. Then $p(C_m) = 2$ with the paths $v_1 v_2 \dots v_{m-1} v_m$ and $v_m v_1$. By Theorem 15, $p(C_m) = T(L(C_m)) = 2$. Since C_m is unicyclic, $T(C_m) = 2$.

Suppose the graph $G = K_m$ is a complete graph on m vertices. Let $V(K_m) = \{v_1, \dots, v_m\}$ with edges $v_i v_j$ for $i \neq j$ and i, j in $\{1, 2, \dots, m\}$. By Theorem 15, $T(K_m) \leq p(K_m) = T(L(K_m))$. The tree cover number of the complete graph was described in Chapter 3. Since $T(K_m) = \lceil \frac{m}{2} \rceil$ and, by Theorem 17, $p(K_m) = \lceil \frac{m}{2} \rceil$, we have the equality $T(K_m) = T(L(K_m))$. □

Strict inequality in Theorem 16 is possible. This case is discussed below.

Theorem 19 ([19])**.** *If G is any tree, then $p(G) = \frac{k}{2}$, where k is the number of vertices of odd degree.*

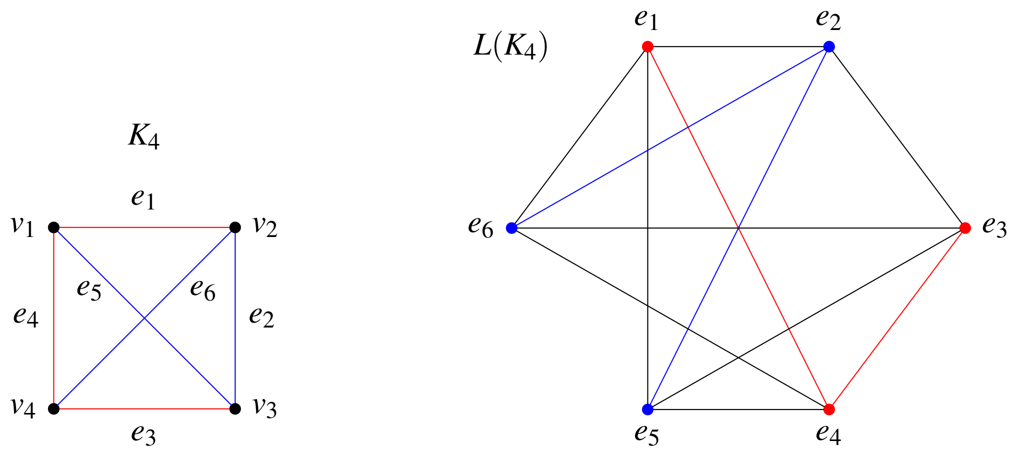


Figure 13. The complete graph K_4 has $p(K_4) = 2$, shown here in red and blue. The paths $v_2v_1v_4v_3$ and $v_1v_3v_2v_4$ in K_4 induce the tree cover $e_1e_4e_3$ and $e_6e_2e_5$ in $L(K_4)$.

Proposition 20. *If G is a tree, then $T(L(G)) = \frac{k}{2}$, where k is the number of vertices of odd degree.*

Proof. From Theorem 19, $p(G) = \frac{k}{2}$, where k is the number of vertices of odd degree. Thus $T(L(G)) = \frac{k}{2}$ using Theorem 15. □

Since for any tree G, $T(G) = 1$, it is easy to see from Proposition 20 that $T(G) < T(L(G))$ when G is a tree that is not a path. An example of this can be seen in Figure 14.

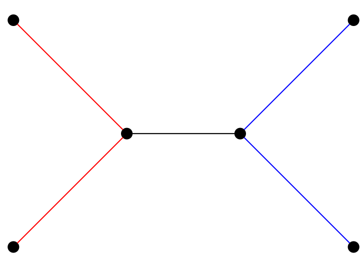


Figure 14. For the above graph G, $p(G) = 3$ shown in colors red, blue, and black. Thus, $T(L(G)) = 3$ but $T(G) = 1$.

Using results on the path decomposition number for complete bipartite and unicyclic graphs, one can find the tree cover number of their line graphs.

Theorem 21 ([14]). *Let $K_{m,n}$ be the complete bipartite graph with $m \leq n$. Then*

$$p(K_{m,n}) = \begin{cases} \frac{m+n}{2} & \textit{if } mn \textit{ is odd}, \\ \lceil \frac{mn}{2m-1} \rceil & \textit{if } mn \textit{ is even and } m \neq n, \\ \lceil \frac{n}{2} \rceil & \textit{otherwise}. \end{cases}$$

Proposition 22. *If $K_{m,n}$ is the complete bipartite graph, $m \leq n$, then*

$$T(L(K_{m,n})) = \begin{cases} \frac{m+n}{2} & \textit{if } mn \textit{ is odd}, \\ \lceil \frac{mn}{2m-1} \rceil & \textit{if } mn \textit{ is even and } m \neq n, \\ \lceil \frac{n}{2} \rceil & \textit{otherwise}. \end{cases}$$

Proof. From Theorem 21,

$$p(K_{m,n}) = \begin{cases} \frac{m+n}{2} & \text{if } mn \text{ is odd}, \\ \lceil \frac{mn}{2m-1} \rceil & \text{if } mn \text{ is even and } m \neq n, \\ \lceil \frac{n}{2} \rceil & \text{otherwise}. \end{cases}$$

This is exactly equal to $T(L(K_{m,n}))$ using Theorem 15. □

Theorem 23 ([1]). *Let G be a unicyclic graph with cycle C. Let r denote the number of vertices of degree greater than two on C. Let k denote the number of vertices of odd degree. Then*

$$p(G) = \begin{cases} 2 & \textit{if } r = 0, \\ \frac{k}{2} + 1 & \textit{if } r = 1, \\ \frac{k}{2} & \textit{otherwise}. \end{cases}$$

Proposition 24. *If G is a unicyclic graph with cycle C, such that r is the number of vertices with degree greater than two on C and k is the number of vertices of odd degree, then*

$$T(L(G)) = \begin{cases} 2 & \text{if } r = 0, \\ \frac{k}{2} + 1 & \text{if } r = 1, \\ \frac{k}{2} & \text{otherwise.} \end{cases}$$

Proof. From Theorem 23,

$$p(G) = \begin{cases} 2 & \text{if } r = 0, \\ \frac{k}{2} + 1 & \text{if } r = 1, \\ \frac{k}{2} & \text{otherwise.} \end{cases}$$

Using Theorem 15, this is exactly equal to $T(L(G))$. □

V.2. Maximum Semidefinite Nullity of Line Graphs

It has been shown in [18] that if G is a simple graph with $|G| \geq 2$ and G contains a Hamilton path, then $msr(L(G)) = |G| - 2$. In this case, $M_+(L(G)) = |E(G)| - |V(G)| + 2$.

For example, if $G = P_m$, then $M_+(L(P_m)) = (m-1) - m + 2 = 1$ showing $T(L(P_m)) = 1 = M_+(L(P_m))$. If $G = C_m$, then $M_+(L(C_m)) = m - m + 2 = 2$. In this case, also, $T(L(C_m)) = 2 = M_+(L(C_m))$.

It has been shown in [4, 18] that $msr(L(K_n)) = n - 2$. Hence $M_+(L(K_n)) = \frac{n(n-1)}{2} - n + 2$. It is easy to verify that $T(L(K_n)) = \lceil \frac{n}{2} \rceil \leq \frac{n(n-1)}{2} - n + 2 = M_+(L(K_n))$ (as the line intersects the parabola at $(2, 1)$ and stays below otherwise or can be justified algebraically).

CHAPTER VI

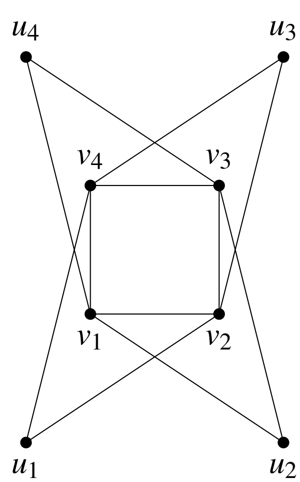
TREE COVER NUMBER OF SHADOW GRAPHS

In this chapter we find the tree cover number of the shadow graph and p-shadow graph for paths, cycles, complete graphs, and complete bipartite graphs. We show that for shadow graphs and p-shadow graphs of paths, the maximum semidefinite nullity is equal to the tree cover number.

VI.1. Tree Cover Number of Shadow Graphs

Let us first give the definition of a shadow graph $S(G)$ of a given graph G. Assume G is a connected simple graph and $|G| = n \geq 2$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. In $S(G)$, we add n shadow vertices to the vertex set of G so that $S(G)$ has the vertex set $\{v_1, u_1, v_2, u_2, \ldots, v_n, u_n\}$. The edge set of $S(G)$ consists of $E(G)$ along with new edges joining each shadow vertex u_i to the neighbors of v_i for $i = 1, 2, \ldots, n$.

As an example, if G is C_4, then $S(G)$ is presented as follows:



Observation 25. *Let G be a simple graph. If $|G| = n \geq 3$, then $S(G)$ has a cycle. For example, a shadow vertex u of a vertex v along with two neighbors x and y of v creates a cycle in $S(G)$. Hence the tree cover number of a shadow graph must be at least 2.*

One could extend the definition of a shadow graph to a p-shadow graph ($p > 1$), denoted $S_p(G)$ where, for each vertex v_i of G, add p shadow vertices $u_i^1, u_i^2, \ldots, u_i^p$ and join each of them to the neighbors of v_i in G.

Theorem 26. *Let P_m be a path on m vertices with $m \geq 3$. Then $T(S(P_m)) = 2$. For $p > 1$, $T(S_p(P_m)) = \lceil \frac{m}{2} \rceil$.*

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and P_m has edges $v_i v_{i+1}$ for $i = 1, \ldots, m-1$. Let $u_1, u_2, \ldots, u_m$ be the shadow vertices of $v_1, v_2, \ldots, v_m$, respectively.

- Case 1: m is even.

In this case $v_1 u_2 v_3 u_4 \ldots v_{m-1} u_m$ and $u_1 v_2 u_3 v_4 \ldots u_{m-1} v_m$ are two vertex disjoint induced paths in $S(P_m)$ which cover all the vertices of $S(P_m)$. See Figure 15 for an example. Hence $T(S(P_m)) \leq 2$. Since $m \geq 3$, by Observation 25, $T(S(P_m)) \geq 2$.

- Case 2: m is odd.

The paths $v_1 u_2 v_3 u_4 \ldots u_{m-1} v_m$ and $u_1 v_2 u_3 v_4 \ldots v_{m-1} u_m$ are induced in $S(P_m)$ and are vertex disjoint. As in the previous case, we conclude $T(S(P_m)) = 2$.

Now we consider $S_p(P_m)$ for $p > 1$.

- Case 1: m is even.

In this case, let T_i be the tree induced by the vertices $\{v_i v_{i+1} u_i^1 \ldots u_i^p u_{i+1}^1 \ldots u_{i+1}^p\}$ where $i = 1, 3, \ldots, m-1$. See Figure 16 for an example. These are $\frac{m}{2}$ vertex disjoint induced trees in $S_p(P_m)$ that cover all the vertices. Hence $T(S_p(P_m)) \leq \frac{m}{2}$.

- Case 2: m is odd.

In this case for $i = 1, 3, 5, \ldots, m-4$ we use the same trees described in the previous case. There are $\frac{m-3}{2}$ such trees. In addition we consider the two trees induced by

$$\{v_{m-2} v_{m-1} u_{m-2}^1 \ldots u_{m-2}^p u_{m-1}^1 \ldots u_{m-1}^p u_m^1 \ldots u_m^p\}$$

and by the single vertex v_m. These $\frac{m-3}{2} + 2 = \frac{m+1}{2} = \lceil \frac{m}{2} \rceil$ vertex disjoint induced trees cover all the vertices of $S_p(P_m)$. Hence $T(S_p(P_m)) \leq \lceil \frac{m}{2} \rceil$.

We show that every tree in a minimal tree cover cannot contain more than two adjacent vertices of the path P_m. Suppose $v_{i-1}v_iv_{i+1}$ is a subpath induced in a tree of the tree cover. Then all the shadow vertices $u_i^1, \ldots, u_i^p$ have to be isolated vertices in the tree cover as otherwise a cycle would be induced by any one of those shadow vertices along with $v_{i-1}v_iv_{i+1}$. Hence the number of trees in the tree cover is $\lceil \frac{m}{3} \rceil + p\lfloor \frac{m}{3} \rfloor \geq (1+p)\lfloor \frac{m}{3} \rfloor \geq 3\lfloor \frac{m}{3} \rfloor \geq \lceil \frac{m}{2} \rceil$. By a similar argument, there will be $\lfloor \frac{k}{2} \rfloor p \lfloor \frac{m}{k} \rfloor$ isolated vertices to be covered if we consider induced paths of length $k \geq 3$ in a tree of the tree cover. Therefore, the number of trees in the tree cover is

$$\left\lceil \frac{m}{k} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor p \left\lfloor \frac{m}{k} \right\rfloor \geq \left(1 + p\left\lfloor \frac{k}{2} \right\rfloor\right) \left\lfloor \frac{m}{k} \right\rfloor \geq \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{m}{k} \right\rfloor \geq \left\lceil \frac{m}{2} \right\rceil.$$

Hence $\lceil \frac{m}{2} \rceil$ is the minimum number of trees in the tree cover. Therefore, we conclude that $T(S_p(P_m)) = \lceil \frac{m}{2} \rceil$. □

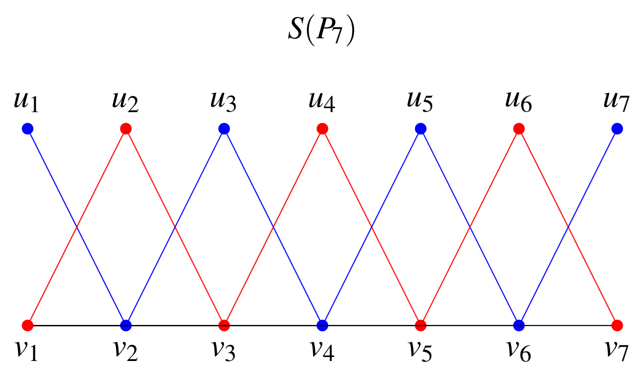


Figure 15. The graph $S(P_7)$ has tree cover number $T(S(P_7)) = 2$. The two vertex disjoint induced trees are represented in red and blue.

Theorem 27. *Let St_m be a star on m vertices with $m \geq 3$. Then $T(S_p(St_m)) = 2$ for all $p \geq 1$.*

Proof. Let $V(St_m) = \{v_1, v_2, \ldots, v_m\}$ and let the star have edges v_1v_i for $i = 2, \ldots, m$. We let $u_1^j, u_2^j, \ldots, u_m^j$ for $j = 1, \ldots, p$ be the shadow vertices. Consider the stars induced by

$$\{v_1, v_2, u_2^1, \ldots, u_2^p, v_3, u_3^1, \ldots, u_3^p, \ldots, v_{m-1}, u_{m-1}^1, \ldots, u_{m-1}^p, u_m^1, \ldots, u_m^p\}$$

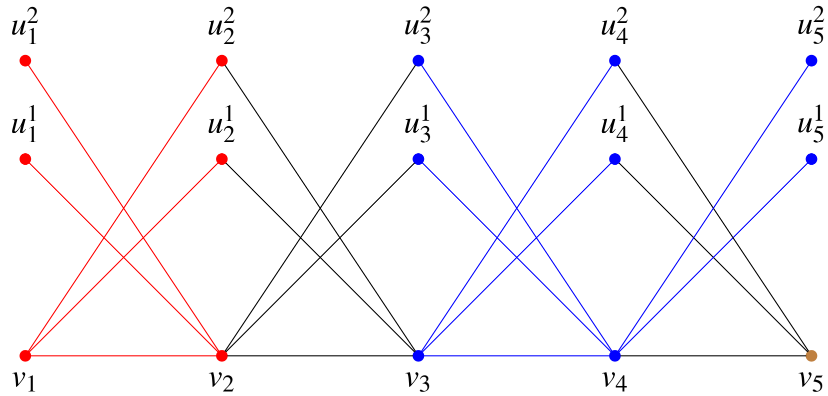


Figure 16. The graph of $S_2(P_5)$ has tree cover number $T(S_2(P_5)) = 3$. The three vertex disjoint induced trees are represented in red, blue, and brown.

and $\{v_m, u_1^1, \ldots, u_1^p\}$. See Figure 17 for an example. These are two vertex disjoint induced trees which cover all of the vertices of $S_p(St_m)$. Hence $T(S(St_m)) \leq 2$. Since $m \geq 3$, by Observation 25, $T(S_p(St_m)) \geq 2$. $\square$

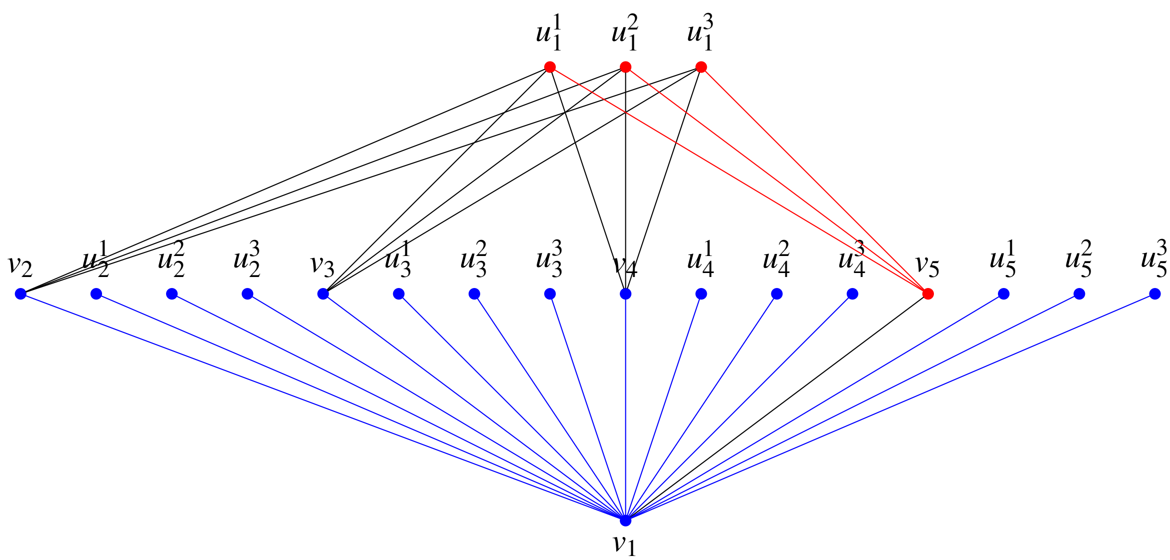


Figure 17. The graph $S_3(St_5)$ has tree cover number $T(S_3(St_5)) = 2$. The two vertex disjoint induced trees are represented in red and blue.

Theorem 28. *Let C_m be a cycle on m vertices with $m \geq 3$. Then*

$$T(S(C_m)) = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 3 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let $V(C_m) = \{v_1, v_2, \ldots, v_m\}$, and the cycle has edges v_1v_m and v_iv_{i+1} for $i = 1, \ldots, m-1$. Let $u_1, \ldots, u_m$ be the shadow vertices.

- Case 1: m is even.

In this case, the graphs induced by

$$\{u_1, v_1, v_2, u_3, v_4, u_5, \ldots, u_{m-1}\} \text{ and } \{u_2, v_3, u_4, v_5, \ldots, v_{m-1}, v_m, u_m\}$$

are two vertex disjoint induced trees which cover all the vertices of $S(C_m)$. This shows $T(S(C_m)) \leq 2$. Since $m \geq 3$, by Observation 25, $T(S(C_m)) \geq 2$.

- Case 2: m is odd.

The subgraphs induced by

$$\{u_1, v_2, u_3, v_4, \ldots, u_{m-2}, v_{m-1}, u_m\} \text{ and } \{u_2, v_3, u_4, v_5, \ldots, u_{m-1}, v_m\}$$

as well as the isolated vertex v_1 are three vertex disjoint induced paths which cover all the vertices of $S(C_m)$. See Figure 18 for an example. Hence $T(S(C_m) \leq 3$. Since $m \geq 3$ by Observation 25, $T(S(C_m)) \geq 2$. It remains to show $T(S(C_m)) \neq 2$.

By way of contradiction, assume there exist two vertex disjoint induced trees T_1 and T_2 that cover the vertices of $S(C_m)$. Since $|S(C_m)| = 2m$, we assume without loss of generality T_1 is induced on at least m vertices. As observed in the proof of Theorem 26, suppose $v_{i-1}v_iv_{i+1}$ is a path induced in T_1. Then u_i becomes an isolated vertex as otherwise a cycle would be induced by $\{u_i, u_{i-1}, v_i, v_{i+1}\}$. This increases the number of trees in the tree cover number to more than two, contradicting our assumption. Since only two adjacent vertices of C_m can be included in T_1, we label them v_1 and v_2. The maximum number of vertices in an induced tree T_1 is m and the tree is induced by $\{u_1, v_1, v_2, u_3, v_4, v_5, \ldots, u_{m-2}, v_{m-1}\}$. The graph induced in $S(C_m)$ by the remaining vertices has two connected components. Thus, three vertex disjoint, induced trees are required to cover $V(S(C_m))$. □

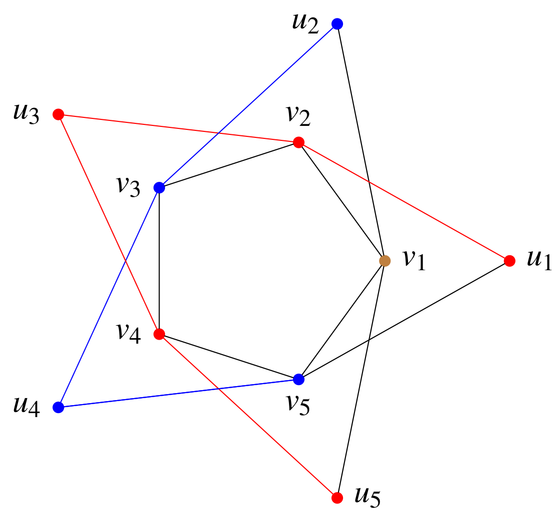


Figure 18. The graph $S(C_5)$ has tree cover number $T(S(C_5)) = 3$. The three vertex disjoint induced trees are represented in red, blue, and brown.

Theorem 29. *Let K_m be a complete graph on m vertices. Then*

$$T(S(K_m)) = \begin{cases} \lceil \frac{m}{2} \rceil & \text{if } m \text{ is even,} \\ \lceil \frac{m}{2} \rceil + 1 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $E(K_m) = \{v_i v_j : i \neq j\}$ for $i, j = 1, \ldots, m$. Let $u_1, \ldots, u_m$ be the shadow vertices of $v_1, v_2, \ldots, v_m$. Note that any three vertices of K_m induce a cycle. Thus any tree in the tree cover of $S(K_m)$ cannot include more than two vertices v_i, v_j for $i \neq j$ and $i, j \in \{1, \ldots, m\}$. Hence, $\lceil \frac{m}{2} \rceil \leq T(S(K_m))$.

Consider any minimal tree cover of K_m, when m is even. This minimal tree cover consists of $\frac{m}{2}$ edges of K_m. Thus $T(K_m) = \frac{m}{2}$. Now in $S(K_m)$ consider the $\frac{m}{2}$ trees induced by

$\{v_i, v_{i+1}, u_i, u_{i+1}\}$, for $i = 1, 3, \ldots, m-1$. These trees cover all the vertices of $S(K_m)$. Therefore $T(S(K_m)) = \frac{m}{2} = \lceil \frac{m}{2} \rceil$.

Consider any minimal tree cover of K_m, when m is odd. This minimal tree cover consists of $\lfloor \frac{m}{2} \rfloor$ edges of K_m and a single vertex denoted as v_1. Thus $T(K_m) = \lceil \frac{m}{2} \rceil$. Now in $S(K_m)$ consider the $\lfloor \frac{m}{2} \rfloor$ trees induced by $\{v_i, v_{i+1}, u_i, u_{i+1}\}$, for $i = 2, \ldots, m-1$. These trees cover all the vertices except for v_1 and u_1. Now v_1 and u_1 have to be isolated vertices in the tree cover of $S(K_m)$ because $v_i v_j v_1$ or $v_i v_j u_1$ for $i \neq j$, $i, j \neq 1$ induce a cycle. Therefore $T(S(K_m)) = \lfloor \frac{m}{2} \rfloor + 2 = \lceil \frac{m}{2} \rceil + 1$. $\square$

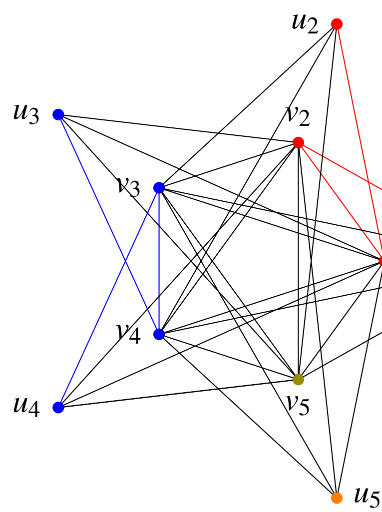


Figure 19. The graph $S(K_5)$ has tree cover number $T(S(K_5)) = \lceil \frac{5}{2} \rceil + 1 = 4$. The three vertex disjoint induced trees are represented in red, blue, olive, and orange.

Theorem 30. *Let $K_{m,n}$ be a complete bipartite graph on mn vertices. Then $T(S(K_{m,n})) = 2$.*

Proof. Let $V(K_{m,n}) = \{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\}$ where $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are independent sets. The complete bipartite graph has edges $v_i w_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $u_1, u_2, \ldots, u_m$ be the shadow vertices of $v_1, \ldots, v_m$ and $y_1, \ldots, y_n$ be the shadow vertices of $w_1, \ldots, w_n$. The shadow graph has additional edges $u_i w_j$ and $v_i y_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots n$. The stars $v_1 y_1 \ldots y_n w_1 w_2 \ldots w_{n-1}$ and $w_n v_2 v_3 \ldots v_m u_1 \ldots u_m$ are vertex disjoint induced trees that cover all the vertices of $S(K_{m,n})$. Hence $T(S(K_{m,n})) \leq 2$. By Observation 25, $T(S(K_{m,n})) \geq 2$. $\square$

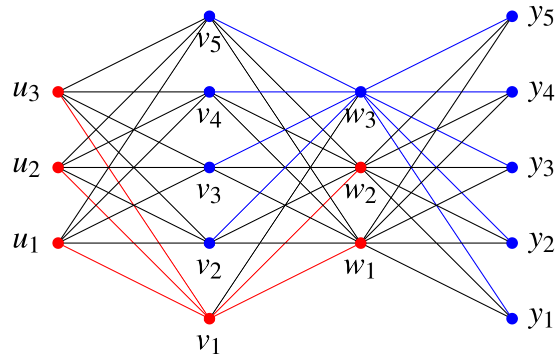


Figure 20. The graph of $S(K_{5,3})$ has tree cover number $T(S(K_{5,3})) = 2$. The two vertex disjoint induced trees are represented in red and blue.

VI.2. Maximum Semidefinite Nullity of Shadow Graphs

1) $T(S(P_m)) = M_+(S(P_m))$: Consider the graph $S(P_m)$. Since each shadow vertex has degree at most two, we orthogonally remove $u_1, u_2, \ldots, u_m$ sequentially. Each degree two shadow vertex will create a triangle consisting of only single edges. The resulting graph is chordal and the minimum semidefinite rank is the clique cover number. The minimum number of triangles needed to cover all the edges is equal to $m-2$. To this, we add m, from orthogonal removal of $u_1, u_2, \ldots, u_m$, to get the minimum semidefinite rank of $S(P_m)$ equal to $2m-2$. Therefore, $M_+(S(P_m)) = 2m-(2m-2) = 2$. Hence $M_+(S(P_m)) = T(S(P_m))$ from Theorem 26.

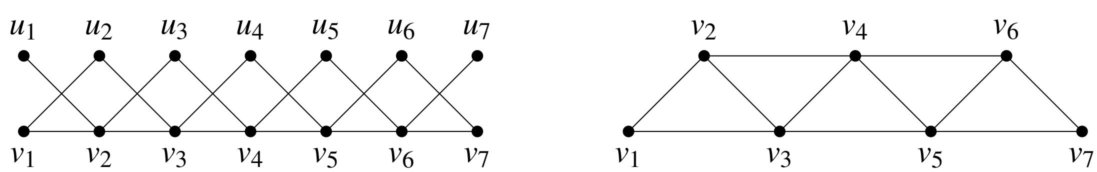


Figure 21. $S(P_7)$ and chordal graph after orthogonal removal.

2) $T(S_p(P_m)) = M_+(S_p(P_m))$: Since each shadow vertex has degree at most two, we can sequentially orthogonally remove u_i^j for a fixed i, and $j = 1, \ldots, p$. This will not result in any

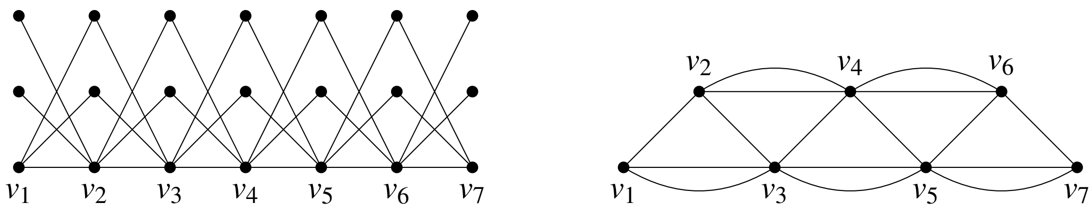


Figure 22. $S_2(P_7)$ and chordal multigraph after orthogonal removal

additional edges for u_i^j, $i = 1, m$, since they are all pendant vertices. For the other shadow vertices that all have degree two, new edges have to be drawn between v_{i-1} and v_{i+1} for $i = 2, \ldots, m-1$. These new edges produce a graph consisting of $m-2$ triangles. This graph is chordal. The minimum semidefinite rank of the graph is equal to its clique cover number. In this instance, we count the minimum number of triangles needed to cover all the single edges in the graph. This number is $\lfloor \frac{m}{2} \rfloor$. To this, we add the number of vertices of degree at most two that were orthogonally removed. Therefore, the minimum semidefinite rank is $pm + \lfloor \frac{m}{2} \rfloor$. The maximum nullity of $S_p(P_m)$ is equal to $(pm+m) - (pm + \lfloor \frac{m}{2} \rfloor) = m - \lfloor \frac{m}{2} \rfloor = \lceil \frac{m}{2} \rceil$. Hence $M_+(S_p(P_m)) = T(S_p(P_m))$ from Theorem 26 for $p \geq 1$.

CHAPTER VII

TREE COVER NUMBER OF CORONA OF TWO GRAPHS

In this chapter, we give an upper bound on the tree cover number of the corona of two simple connected graphs. We also provide a case where equality holds. In addition, we show that the conjecture holds for the corona when each of the two graphs individually satisfy the conjecture.

VII.1. Tree Cover Number of Corona Graphs

The corona $G_1 \circ G_2$ of two simple connected graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 and $|G_1|$ copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. It follows from the definition of the corona that $G_1 \circ G_2$ has $|G_1| + |G_1||G_2|$ vertices and $m_1 + |G_1|m_2 + |G_1||G_2|$ edges where m_1 and m_2 are the sizes of G_1 and G_2 respectively.

Next we give an upper bound on $T(G_1 \circ G_2)$ and show that in some cases the equality holds.

Proposition 31. *Let G and H be simple connected graphs. Then $T(G \circ H) \leq T(G) + |G|T(H)$.*

Proof. Let $T_1, T_2, \dots, T_k$ be a minimal tree cover of G and $M_1, M_2, \dots, M_l$ be a minimal tree cover of H. If we make $|G|$ copies of $M_i, 1 \leq i \leq l$, then these trees together with $T_1, T_2, \dots, T_k$ form a tree cover of $G \circ H$. Hence $T(G \circ H) \leq T(G) + |G|T(H)$. □

When $H = P_m$ in the previous proposition equality holds.

Theorem 32. *Let G be a simple graph and P_m is a path. Then $T(G \circ P_m) = T(G) + |G|$.*

Proof. Let $\{T_1, \dots, T_k\}$ be a minimal tree cover of G. If each of the $|G|$ paths cover the vertices of the path then $T(G \circ P_m) \leq T(G) + |G|$. Let v_i be a vertex of G covered by a tree $T_l, 1 \leq l \leq k$. Let $\{w_1^i, w_2^i, \dots, w_m^i\}$ be the vertices of the path P_m adjacent to v_i. If we were to get a smaller tree cover number for $G \circ P_m$, we should be able to extend T_l to cover the vertices of the path adjacent to v_i. Two vertices adjacent in the path together with v_i will induce a cycle. Hence T_l cannot be extended to include adjacent vertices in the path P_m^i. If T_l is extended to include the end vertices of P_m^i, then $T(G)+|G|$ is the size of the tree cover of $G \circ P_m$ as a path is needed to cover the remaining vertices of P_m^i. Any other option of including vertices of P_m^i to T_l would require more than one path to cover the remaining vertices of P_m^i. Therefore the minimal tree cover has $T(G)+|G|$ trees. Hence $T(G \circ P_m) = T(G)+|G|$. □

Corollary 33. *Let P_m and P_n be simple paths. Then $T(P_n \circ P_m) = n+1$.*

Proof. Take $G = P_n$ in the previous theorem. □

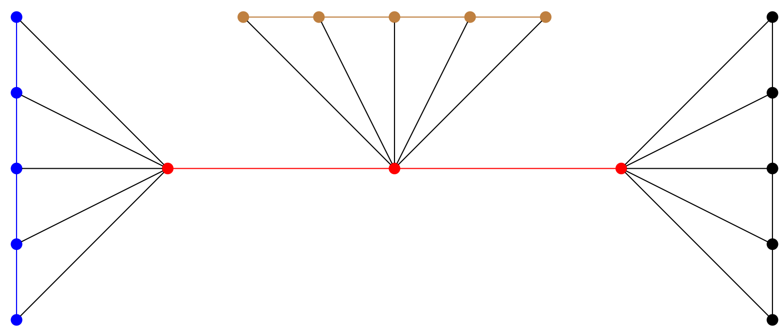


Figure 23. Above is the graph of $P_3 \circ P_5$. The induced trees are red, blue, brown, and black.

Remark 34. *It is clear that $T(P_m \circ P_n) = m+1$. When $m \neq n$ we see that $T(P_m \circ P_n) \neq T(P_n \circ P_m)$. In general, $T(G \circ H) \neq T(H \circ G)$. For example, when $G = K_4$ and $H = P_4$, the above theorem shows $T(K_4 \circ P_4) = 6$ and a calculation shows $T(P_4 \circ K_4) = 9$.*

In Figure 24, we provide an example where $T(G \circ H) < T(G)+|G|T(H)$.

VII.2. Maximum Semidefinite Nullity of Corona Graphs

Using Theorem 8 on the vertex sum of two graphs, we can show that $msr(G \circ H) = msr(G)+|G|msr(H)$. From this we deduce $M_+(G \circ H) = M_+(G)+|G|M_+(H)$. Thus we have the following proposition.

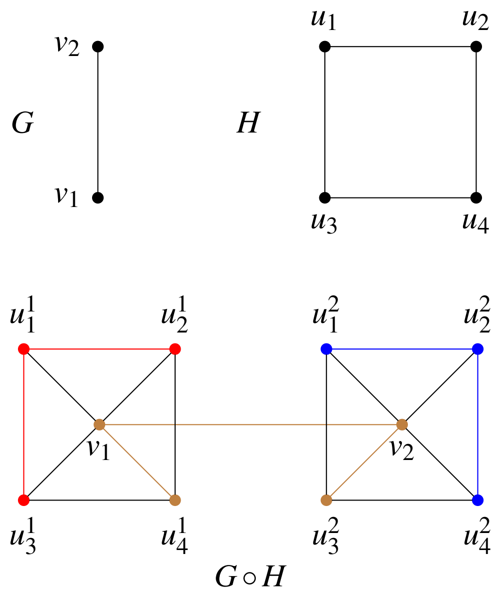


Figure 24. Here, $V(G) = \{v_1, v_2\}$ and $V(H) = \{u_1, u_2, u_3, u_4\}$. The corona (given below) has $T(G \circ H) = 3 < 5 = T(G) + |G|T(H)$. The three induced trees are brown, red, and blue.

Proposition 35. *If $T(G) \leq M_+(G)$ and $T(H) \leq M_+(H)$ then $T(G \circ H) \leq M_+(G \circ H)$.*

Proof. From Proposition 31, we know that $T(G \circ H) \leq T(G) + |G|T(H)$. From $msr(G \circ H)$, we deduced that $M_+(G \circ H) = M_+(G) + |G|M_+(H)$. Therefore if G and H satisfy $T(G) \leq M_+(G)$ and $T(H) \leq M_+(H)$ then $T(G \circ H) \leq T(G) + |G|T(H) \leq M_+(G) + |G|M_+(H) = M_+(G \circ H)$. □

In particular, $T(P_n \circ P_m) = n + 1 = M_+(P_n \circ P_m)$.

CHAPTER VIII

TREE COVER NUMBER OF THE PRODUCT OF TWO GRAPHS

In this chapter, we consider the tree cover number of the cartesian product and lexicographic product of two simple connected graphs. We find the tree cover number of numerous types of cartesian products and confirm the conjecture $T(G) \leq M_{+}(G)$ holds. We also find the tree cover number of the lexicographic product of two paths. These results show that the two different products affect the tree cover number greatly.

VIII.1. Tree Cover Number of Cartesian Products

The *cartesian product* of simple graphs G_1 and G_2 is the graph $G_1 \square G_2$ whose vertex set is $V(G_1) \times V(G_2)$ and whose edge set is the set of all pairs $\{(u_1,v_1),(u_2,v_2)\}$ such that $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$. Thus, for each edge u_1u_2 of G_1 and each edge v_1v_2 of G_2 there are four edges in $G_1 \square G_2$ (see Figure 25 below) [7].

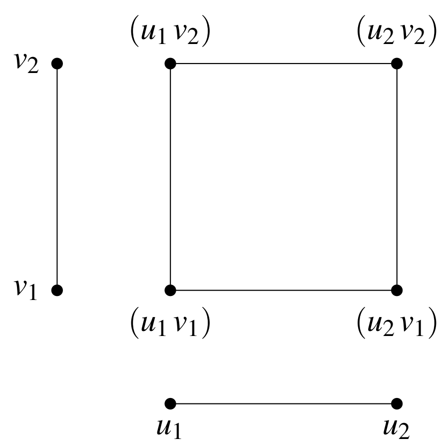


Figure 25. Let G_1 be the graph on vertices $\{v_1,v_2\}$ and G_2 be the graph on vertices $\{u_1,u_2\}$. Then $G_1 \square G_2$ is the four-cycle, seen above.

More generally, the cartesian product $P_m \square P_n$ of two paths P_m and P_n is the $(m \times n)$-grid.

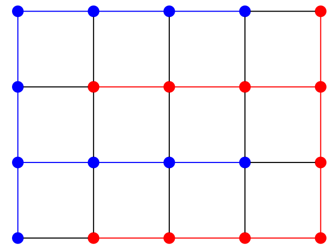


Figure 26. $P_5 \square P_4$ can be covered by two vertex disjoint induced trees. These trees are shown in red (T_1) and blue (T_2).

Proposition 36. *Let P_m and P_n be paths on m vertices and n vertices, respectively. Then*

$$T(P_m \square P_n) = 2.$$

Proof. Let T_1 be the (red) tree induced in $P_m \square P_n$ by the vertices in the right most column of the $m \times n$ grid along with vertices in the odd numbered rows that are *not* in the left most column of the grid. Let T_2 be the (blue) tree induced by the remaining vertices of $P_m \square P_n$. These are vertex disjoint simple induced trees in $P_m \square P_n$. Therefore $T(P_m \square P_n) \leq 2$. Clearly, $P_m \square P_n$ has induced cycles. Hence $T(P_m \square P_n) > 1$. Therefore, $T(P_m \square P_n) = 2$. □

Proposition 37. *Let C_m be a cycle on m vertices and P_n be a path on n vertices. Then*

$$T(C_m \square P_n) = 2.$$

Proof. The two trees described in Proposition 36 cover the vertices of $C_m \square P_n$. Since the vertices of the first column are in one tree and the vertices of the last column are in another tree, a cycle C_m cannot be induced. Thus $T(C_m \square P_n) = 2$. □

We can extend the result of Proposition 36 to $\tau_m \square \tau_n$ where τ_m (respectively τ_n) is a tree on m (respectively n vertices).

Theorem 38. *Let τ_m and τ_n be trees on m and n vertices respectively. Then $T(\tau_m \square \tau_n) = 2$.*

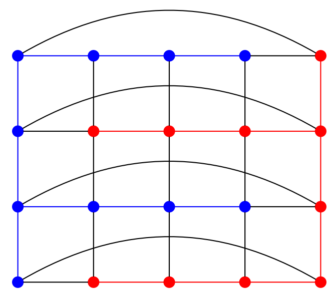


Figure 27. $C_5 \square P_4$ can be covered by two vertex disjoint induced trees. These trees are shown in red and blue.

Proof. Let $V(\tau_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(\tau_n) = \{u_1, u_2, \ldots, u_n\}$. The vertices of $\tau_m \square \tau_n$ are $\{(v_i, u_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. Since every tree must have at least two pendant vertices, we consider a longest path in τ_m with endpoints a and b. We label τ_m such that $v_1 = a$ and $v_m = b$. Since any tree is bipartite, we can partition the vertices of τ_n into two partite sets, H_1 and H_2. Let G_1 be the tree induced by the vertices in the right-most column of the $m \times n$ grid along with the vertices in the i-th rows, where $u_i \in H_1$, that are *not* in the left-most column of the grid. Let G_2 be the tree induced by the remaining vertices of $\tau_m \square \tau_n$. No cycles are induced between vertices in distinct rows of G_1 (or G_2) because of the vertices (v_i, u_j), (v_i, u_k) in G_1, u_j and u_k are in the same partite set. No cycles are induced between vertices in distinct columns of G_1 (or G_2) because each row represents the tree τ_m which is acyclic. Therefore, $T(\tau_m \square \tau_n) \leq 2$. Since $T(\tau_m \square \tau_n) > 1$, the theorem holds. $\square$

Theorem 39. *Let P_m be a path on m vertices and K_n be a complete graph on n vertices. Then $T(K_n \square P_m) = \lceil \frac{n}{2} \rceil$.*

Proof. The nm vertices (v_i, w_j) of $K_n \square P_m$ consists of the vertices $\{v_1, \ldots, v_n\}$ of K_n and the vertices $\{w_1, \ldots, w_m\}$ of P_m. Each row of $K_n \square P_m$ corresponds to K_n and each column corresponds to P_m. Each induced tree in $K_n \square P_m$ contains at most two vertices from each row as otherwise a cycle will be induced. Thus $T(K_n \square P_m) \geq \lceil \frac{n}{2} \rceil$. To prove equality, it suffices to describe a tree cover of size $\lceil \frac{n}{2} \rceil$.

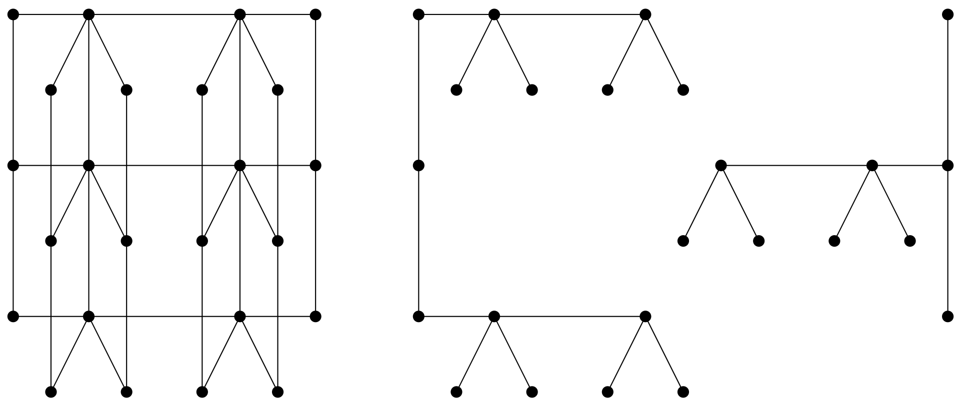


Figure 28. The above graph illustrates Theorem 38. The graph $\tau_8 \square P_3$ can be covered by two vertex disjoint induced trees, displayed as subgraphs to the right of $\tau_8 \square P_3$.

- Case 1: n is even.

Let T_i be the induced graph on vertices

$$\{(i,1),(i+1,1),(i+1,2),(i+2,2),\ldots,(i+m-1 \pmod n, m),(i+m \pmod n, m)\}$$

for $i = 1,3,5,\ldots,n-1$. Then $T_1, T_3, \ldots, T_{n-1}$ are $\frac{n}{2}$ vertex disjoint simple trees which cover $K_n \square P_m$.

- Case 2: n is odd.

In the first $n-1$ columns, cover the vertices as above in the even case. Cover the vertices of the n-th column using a path. Hence $T(K_n \square P_m) \leq \frac{n-1}{2} + 2 = \lceil \frac{n}{2} \rceil$. $\square$

VIII.2. Maximum Semidefinite Nullity of Cartesian Products

From Chapter 4, we know that $msr(G) = |G| - 1$ if and only if G is a tree. Therefore, $M_+(G) = 1$ if and only if G is a tree. Since $P_m \square P_n$, $C_m \square P_n$ and $T_n \square P_m$ are not tree graphs, their maximum semidefinite nullity is at least two. Therefore, $T(P_m \square P_n) \leq M_+(P_m \square P_n)$, $T(C_m \square P_n) \leq M_+(C_m \square P_n)$ and $T(T_n \square P_m) \leq M_+(T_n \square P_m)$. From [4, 18], we know that for $n \geq 2$, $M_+(K_n \square P_m) = n$. Therefore, $T(K_n \square P_m) = \lceil \frac{n}{2} \rceil < n = M_+(K_n \square P_m)$.

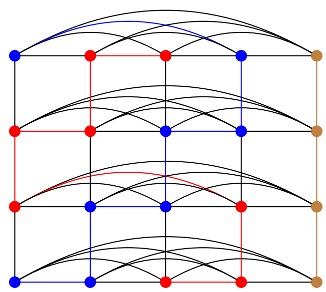


Figure 29. $K_5 \square P_4$ can be covered by three vertex disjoint induced trees. These trees are shown in red, blue, and brown.

VIII.3. Tree Cover Number of Lexicographic Product

Let G and H be simple connected graphs with vertex sets $V(G)$ and $V(H)$, respectively. The lexicographic product of G and H has the vertex set $V(G) \times V(H)$. The edge set consists of those edges joining (u,v) and (u',v') as follows: either $(uu' \in E(G))$ or $(u = u'$ and $vv' \in E(H))$. The lexicographic product of G and H is denoted $G[H]$.

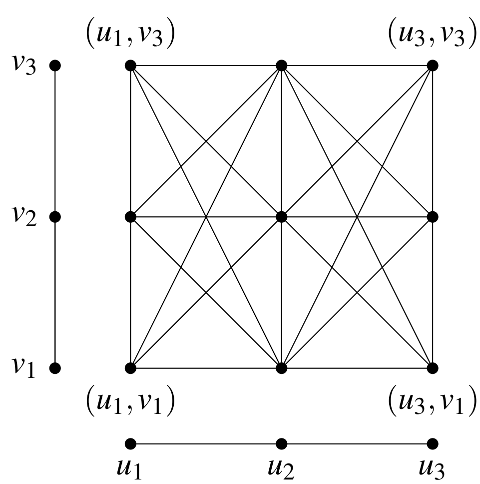


Figure 30. Let G_1 be the path on vertices $\{u_1, u_2, u_3\}$ and G_2 be the path on vertices $\{v_1, v_2, v_3\}$. Then $G_1[G_2]$ is given above.

Proposition 40. *Let P_m and P_n be simple paths on m and n vertices, respectively. Then*

$$T(P_m[P_n]) = \min\{m,n\}.$$

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. By definition of lexicographic product, the vertices $V(P_m) \times V(P_n)$ can be arranged in a grid with m rows and n columns indexed by $\{(v_i, u_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$, where $v_i \in V(P_m)$ and $u_j \in V(P_n)$.

A tree cover consisting of m trees can be constructed by letting T_i be the path induced by vertices of the i-th row $\{(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_n)\}$, for $i \in \{1, \ldots, m\}$. A tree cover consisting of n trees can be constructed by letting S_i be the path induced by vertices on the j-th column $\{(v_1, u_j), (v_2, u_j), \ldots, (v_m, u_j)\}$, for $j \in \{1, \ldots, n\}$. Hence $T(P_m[P_n]) \leq \min\{m,n\}$.

Since each vertex in the i-th row is adjacent to every vertex of the $(i+1)$-th row, and each row forms a path, the vertices $(v_i, u_j), (v_{i+1}, u_j), (v_{i+1}, u_j)$ induce a cycle. Thus, if a tree in the tree cover of $P_m[P_n]$ contains vertices from distinct rows, the maximum number of vertices in an induced tree is n. Similarly, if a tree in the tree cover contains vertices from distinct columns, the maximum number of vertices in an induced tree is m. Since $|G| = mn$, $T(P_m[P_n]) \geq \min\{m,n\}$. □

CHAPTER IX

MORE RESULTS ON TREE COVER NUMBER

In this chapter we present other results on the tree cover number. These results include the tree cover number for m-partite graphs, wheel graphs, and gear graphs. We also prove that deleting a vertex v from a graph G can decrease the tree cover number by at most one.

IX.1. Tree Cover Number of Complete m-partite Graphs

Proposition 41. *Let $K_{n_1,n_2,\ldots,n_m}$ be a complete m-partite graph such that $n_i \geq m$ for $1 \leq i \leq m$. Then $T(K_{n_1,n_2,\ldots,n_m}) = m$.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_m}$ be a complete m-partite graph. Assume $n_1, n_2, \ldots, n_m \geq m$. Without loss of generality, assume $n_1 \leq n_2 \leq \cdots \leq n_m$. Denote the m partite sets as $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$ for $i = 1, 2, \ldots, m$.

A tree cover of size n can be constructed as follows:

Let T_1 be the induced star graph with center x_{11} and the leaves the first $n_2 - 1$ vertices of x_2. Let T_2 be the star graph with center x_{2n_2} and the leaves the first $n_3 - 1$ vertices of x_3. Continuing this process the last tree T_n is a star graph with center x_{mn_m} and the leaves $x_{12}, x_{13}, \ldots, x_{1n_1}$ from X_1. See Figure 31 for an example.

Thus $T_1, T_2, \ldots, T_m$ provide a tree cover of size m on G. Hence $T(G) \leq m$. It remains to show that $T(G) \geq m$. Assume that G could be covered with less than m vertex disjoint induced trees. Note that if $k < m$ trees can cover $V(G)$, then by making the induced trees smaller in size $V(G)$ could be covered by $m - 1$ trees.

Assume, by way of contradiction, that $V(G)$ can be covered by $m - 1$ vertex disjoint induced trees. Note that, since G is a complete m-partite graph, any three vertices from three distinct partite sets will induce a cycle. In addition, the vertices $x_{ij}, x_{ik}, x_{rs}, x_{rt}$ induce a four cycle. Thus any induced tree in the tree cover of G must be a star, with center in one partite set and all pendant vertices in another partite set. Since an entire partite set can be covered by one star induced by the vertices $x_{i1}, x_{j1}, x_{j2}, \ldots, x_{jn_j}$, finding a tree cover of size $m-1$ is equivalent to covering three partite sets with two trees. Since each partite set has at least three elements, it is impossible to partition the vertices so that each partition only contains vertices from two distinct sets and does not include a four cycle. Therefore, $T(G) \neq m-1$ and thus $T(G) = m$. $\square$

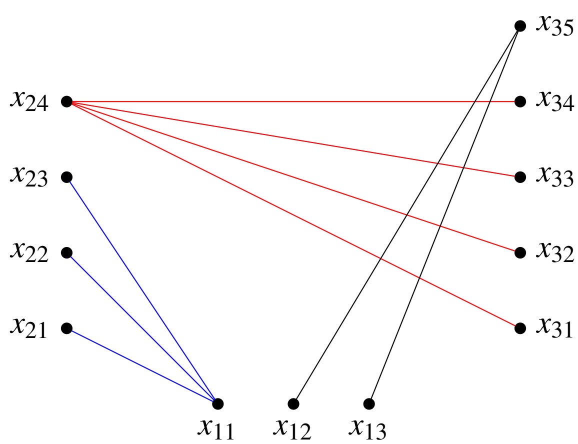


Figure 31. The induced trees in the tree cover of $K_{3,4,5}$ are shown above in blue, red, and black.

IX.2. Maximum Semidefinite Nullity of Complete m-partite Graphs

It has been shown in [13] that for a complete multipartite graph G with nonempty partite sets, $msr(G) = \alpha(G)$. In the proof of Proposition 41, it has been assumed that $K_{n_1,n_2,\ldots,n_m}$ satisfies $n_1 \leq n_2 \leq \cdots \leq n_m$. Therefore, $msr(K_{n_1,n_2,\ldots,n_m}) = n_m$. It has been shown that $T(K_{n_1,n_2,\ldots,n_m}) = m$. Now $M_+(K_{n_1,n_2,\ldots,n_m}) = n_1 + n_2 + \cdots + n_m - n_m = n_1 + n_2 + \cdots + n_{m-1}$. Since $n_1, n_2, \ldots, n_m \geq m$ it follows that $T(K_{n_1,n_2,\ldots,n_m}) \leq M_+(K_{n_1,n_2,\ldots,n_m})$.

IX.3. Wheel Graphs and Gear Graphs

In this section, we find the tree cover number for wheel graphs and gear graphs.

Definition 3. *A wheel graph W_m of order m is a graph that contains an $(m-1)$-cycle for which every vertex of the cycle is connected to one other vertex of the graph.*

Definition 4. *A gear graph G_m is a wheel graph with an extra vertex added between each pair of adjacent vertices of the outer cycle of the wheel. Thus G_m is a graph with order $2m-1$ consisting of a $(2m-2)$-cycle and one other vertex adjacent to the m vertices of the outer cycle.*

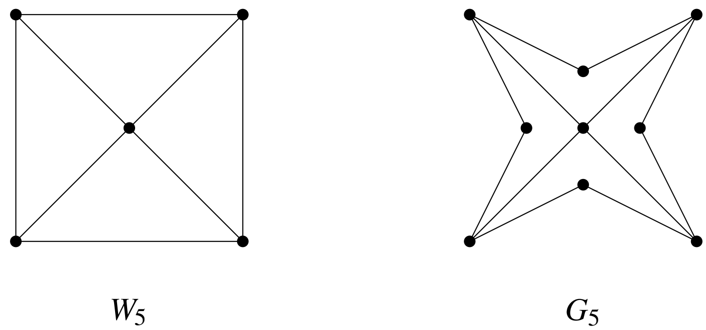


Figure 32. Above are the wheel graph W_5 and gear graph G_5.

Both of these graphs can be covered by two vertex disjoint induced trees.

Proposition 42. *Let W_m be a wheel graph on m vertices. Then $T(W_m) = 2$.*

Proof. Let $V(W_m) = \{v_1, v_2, \ldots, v_m\}$ where v_1 is the center of the wheel graph. Let T_1 be the tree induced in W_m by vertices $\{v_1, v_2\}$ and T_2 be the tree induced in W_m by $\{v_3, \ldots, v_n\}$. Hence $T(W_m) \leq 2$. Since W_m contains cycles, $T(W_m) > 1$. □

Proposition 43. *Let G_m be a gear graph on $2m+1$ vertices. Then $T(G_m) = 2$.*

Proof. Let $V(G_m) = \{v_1, v_2, v_{2,3}, v_3, v_{3,4}, v_4, \ldots, v_{m-1}, v_{m-1,m}, v_m\}$ where v_1 is the center of the gear, and $v_{i,i+1}$ is the vertex that subdivides the edge $v_i v_{i+1}$. Let T_1 be the tree induced by vertices $\{v_1, v_2\}$. Let T_2 be the tree induced by the remaining vertices. Thus $T(G_m) \leq 2$. Since G_m contains cycles, $T(G_m) > 1$. □

IX.4. Maximum Semidefinite Nullity of Wheel Graphs and Gear Graphs

It has been shown in [13] that, for a simple connected graph G on two or more vertices, $msr(G \vee K_1) = msr(G)$. Using this result, we see that

$$msr(W_m) = msr(C_{m-1} \vee K_1) = msr(C_{m-1}) = m-3$$

and $M_+(W_m) = 3$. Thus $T(W_m) < M_+(W_m)$ from Proposition 42. For the gear graph G_m, if we orthogonally remove the $m-1$ degree two vertices subdividing the edges of C_{m-1}, we get a wheel graph W_m. Hence

$$msr(G_m) = (m-1) + msr(W_m) = (m-1) + (m-3) = 2m-4.$$

Thus $M_+(G_m) = (2m-1) - (2m-4) = 3$. From Proposition 43, $T(G_m) < M_+(G_m)$.

IX.5. Vertex Removal

In this section, we consider the effect vertex deletion has on the tree cover number. We find a lower bound on the tree cover number of $G - v$ upon deleting a vertex v from a simple connected graph G. We characterize when the equality holds in this inequality. We give a sufficient condition for tree cover number of vertex deletion to equal the tree cover number of G.

Proposition 44. *Let G be a simple connected graph and v be a vertex in G. Then*

$$T(G) \leq T(G-v) + 1.$$

Proof. Suppose $\mathscr{T} = \{T_1, T_2, \ldots, T_k\}$ is a minimal tree cover of $G - v$. Let $T_{k+1} = \{v\}$. Then $\mathscr{T} \cup \{T_{k+1}\}$ is a tree cover of G. Therefore, $T(G) \leq T(G-v) + 1$. □

The next result characterizes when equality in the above inequality holds.

Theorem 45. *Let G be a simple connected graph and v be a vertex in G. Then $T(G) - 1 = T(G-v)$ if and only if $\{v\}$ is one of the trees in a minimal tree cover of G.*

Proof. ($\Leftarrow$) If $T_1 = \{v\}$ is a tree in a minimal tree cover $\{T_1, T_2, \ldots, T_k\}$ of G. Then

$$T(G-v) \leq k-1 = T(G)-1.$$

From Proposition 44, we conclude that $T(G) = T(G-v)+1$.

($\Rightarrow$) Suppose $T(G) = T(G-v)+1$. Let $\{T_1, T_2, \ldots, T_l\}$ be a minimal tree cover of $G-v$. Then by the hypothesis, $\{T_1, T_2, \ldots, T_l, T_{l+1}\}$, where $T_{l+1} = \{v\}$, is a minimal tree cover of G. □

Theorem 46. *Let G be a simple connected graph and v be a vertex in G. Let $\{T_1^i, T_2^i, \ldots, T_k^i\}$ for $i = 1, 2, \ldots, m$ be all of the minimal tree covers of G. Assume $v \in T_1^i$ for $i = 1, \ldots, m$. If $\deg_{T_1^1}(v) = 1$ and $\deg_{T_1^i}(v) \geq 1$ for $i = 2, \ldots, m$, then $T(G) = T(G-v)$.*

Proof. Let $T_1^{1*} = T_1^1 - v$. Then T_1^{1*} is an induced tree in $G-v$ and $\mathscr{T}_1 = \{T_1^{1*}, T_2^1, \ldots, T_k^1\}$ is a tree cover for $G-v$. Therefore, $T(G-v) \leq T(G)$.

From Proposition 44, we know that $T(G) \leq T(G-v)+1$. Since $\deg_{T_1^i}(v) \geq 1$, for $i = 1, \ldots, m$, we conclude that $T(G-v) \leq T(G) < T(G-v)+1$, using Theorem 45. Since tree cover numbers are integers, we conclude that $T(G-v) = T(G)$. □

In general, $T(G-v)$ can be very large compared to $T(G)$. For example, if $G = K_{1,n}$ a star with v the vertex of degree n in G, then $T(G-v) = n$ while $T(G) = 1$.

CHAPTER X

CONCLUSION

In this thesis, we have found the tree cover number for several different types of graphs, including line graphs, shadow graphs, and graph products. We also confirm in these cases that the conjecture, $T(G) \leq M_+(G)$ holds.

We proved the tree cover number of the line graph of a simple connected graph G is exactly equal to the path decomposition of G. Finding this path decomposition is simpler and faster than computing the tree cover number of the line graph. This result allowed us to find the tree cover number of many different types of line graphs using known results on path decomposition.

We found the tree cover number of the shadow graph of paths, cycles, complete graphs, and complete bipartite graphs. In each of these cases, $T(S(G))$ was greater than or equal to $T(G)$. Based on our results we propose the following conjecture.

Conjecture 47. *If G is a simple connected graph, then $T(G) \leq T(S(G))$.*

We also provided an upper bound for the tree cover number of any corona graph. If $T(G) \leq M_+(G)$ and $T(G) \leq M_+(H)$ then we showed that $(G \circ H) \leq M_+(G \circ H)$. Regarding the cartesian product, we identified the tree cover number of products of trees and paths and complete graphs. For each of these products, we showed that the maximum semidefinite nullity is indeed greater than or equal to the tree cover number.

We also considered the effect vertex deletion has on the tree cover number. We provided a lower bound for the tree cover number of $G - v$, where v is a vertex of a simple connected graph G. We provided a necessary and sufficient condition for when this bound is obtained. We also provided a sufficient condition for $T(G-v)$ to equal $T(G)$. Based on many examples, we propose the following two conjectures.

Conjecture 48. *Let G be a simple connected graph and v be a vertex in G. Let $\{T_1^i, T_2^i, \ldots, T_k^i\}$ for $i = 1, 2, \ldots, m$ be all of the minimal tree covers of G. Assume $v \in T_1^i$ for $i = 1, \ldots, m$. Then $T(G) = T(G-v)$ if and only if $\deg_{T_1^1}(v) = 1$ and $\deg_{T_1^i}(v) \geq 1$ for $i = 2, \ldots, m$.*

Conjecture 49. *Let G be a simple connected graph and v be a vertex in G. Let $\{T_1^i, T_2^i, \ldots, T_k^I\}$ for $i = 1, 2, \ldots, m$ be all of the minimal tree covers of G. Assume $v \in T_1^i$ for $i = 1, \ldots, m$. Then $T(G) < T(G - v)$ if and only if $\deg_{T_1^i}(v) \geq 2$ for $i = 1, \ldots, m$.*

A future goal would be to show that $T(G) \leq M_+(G)$ for every graph G. This provides an upper bound $msr(G) \leq |G| - T(G)$ for every graph G.

REFERENCES

[1] S. Arumugam and J. Suresh Suseela, *Acyclic graphoidal covers and path partitions in a graph*, Discrete Mathematics **190** (1998), no. 1-3, 67–77.

[2] Francesco Barioli, Wayne Barrett, Shaun M. Fallat, H. Tracy Hall, Leslie Hogben, Bryan Shader, P. van den Driessche, and Hein van der Holst, *Zero forcing parameters and minimum rank problems*, Linear Algebra and its Applications **433** (2010), no. 2, 401–411.

[3] Francesco Barioli, Shaun M. Fallat, Lon H. Mitchell, and Sivaram K. Narayan, *Minimum semidefinite rank of outerplanar graphs and the tree cover number*, Electronic Journal of Linear Algebra **22** (2011), 10–21.

[4] Jonathan E. Beagley, Sivaram K. Narayan, Eileen L. Radzwion, Sara P. Rimer, Rachael L. Tomasino, Jennifer L. Wolfe, and Andrew M. Zimmer, *On the minimum semidefinite rank of a graph using vertex sums, graphs with* $msr(G) = |G| - 2$*, and the msr of certain graph classes*, NSF-REU Report, Central Michigan University, 2007.

[5] Lowell W. Beineke, *Derived graphs and digraphs*, Beiträge zur Graphentheorie (1968), 17–33.

[6] ———, *Characterizations of derived graphs*, Journal of Combinatorial Theory **9** (1970), no. 2, 129–135.

[7] J. A. Bondy and U. S. R. Murty, *Graph Theory (Graduate Texts in Mathematics)*, Springer, New York, 2008.

[8] Matthew Booth, Philip Hackney, Benjamin Harris, Charles R. Johnson, Margaret Lay, Terry D. Lenker, Lon H. Mitchell, Sivaram K Narayan, Amanda Pascoe, and Brian D Sutton, *On the minimum semidefinite rank of a simple graph*, Linear and Multilinear Algebra **59** (2011), no. 5, 483–506.

[9] Matthew Booth, Philip Hackney, Benjamin Harris, Charles R. Johnson, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, Amanda Pascoe, Kelly Steinmetz, Brian D. Sutton, and Wendy Wang, *On the minimum rank among positive semidefinite matrices with a given graph*, SIAM Journal on Matrix Analysis and Applications **30** (2008), no. 2, 731–740.

[10] Jason Ekstrand, Craig Erickson, Diana Hay, Leslie Hogben, and Jolie Roat, *Note on positive semidefinite maximum nullity and positive semidefinite zero forcing number of partial* 2*-trees*, Electronic Journal of Linear Algebra **23** (2012), 79–87.

[11] Paul Erdös, Michael Saks, and Vera T Sós, *Maximum induced trees in graphs*, Journal of Combinatorial Theory, Series B **41** (1986), no. 1, 61–79.

[12] Shaun M. Fallat and Leslie Hogben, *The minimum rank of symmetric matrices described by a graph: A survey*, Linear Algebra Appl. **426** (2007), 558–582.

[13] Philip Hackney, Benjamin Harris, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, and Amanda Pascoe, *Linearly independent vertices and minimum semidefinite rank*, Linear Algebra and its Applications **431** (2009), no. 8, 1105–1115.

[14] Frank Harary and Allen J. Schwenk, *Evolution of the path number of a graph, covering and packing in graphs II*, Graph Theory and Computing (R.C. Read, ed.) (1972), 39–45.

[15] Roger A. Horn and Charles R. Johnson, *Matrix Analysis*, 2 ed., Cambridge University Press, 2013.

[16] J. Krausz, *Demonstration nouvelle d'une thorme de whitney sur les rseaux*, Math. Fiz. **50** (1943), 75–85.

[17] Torrence D. Parsons and Tomaž Pisanski, *Vector representations of graphs*, Discrete Mathematics **78** (1989), no. 1-2, 143–154.

[18] Travis Anthony Peters, *Positive semidefinite maximum nullity and zero forcing number*, Ph.D. thesis, Doctoral Dissertation, Iowa State University, 2012.

[19] Ralph G. Stanton, D. D. Cowan, and L. O. James, *Some results on path numbers*, Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing, 1970, pp. 112–135.

[20] H. van der Holst, *Graphs whose positive semi-definite matrices have nullity at most two*, Linear Algebra Appl. **375** (2003), 1–11.

[21] A. C. M. van Rooij and H. S. Wilf, *The interchange graph of a finite graph*, Acta Mathematica Hungarica **16** (1965), no. 3-4, 263–269.

[22] Douglas B. West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.