PBW BASES AND MARGINALLY LARGE TABLEAUX IN TYPES B_n AND C_n

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ABSTRACT

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The original development of the infinity crystal $B(\infty)$ involved an elaborate algebraic construction. Since the original papers by Kashiwara on the subject, these descriptions have been adapted to use various combinatorial objects which encode the details of the algebraic constructions. Since all of these constructions represent the same crystal $B(\infty)$, it is desirable to establish crystal-preserving bijections between these combinatorial objects. The two realizations discussed here are described in terms of marginally large tableaux in one case, and Kostant partitions in the other. In this work, for $B(\infty)$ of type B_n or C_n, an explicit description of the isomorphism between the two realizations is given.

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CHAPTER I

INTRODUCTION

The infinity crystal $B(\infty)$ is a combinatorial object, associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$. The structure of $B(\infty)$ contains information about the integrable highest weight representations of $\mathfrak{g}$ and of the quantum group $U_q(\mathfrak{g})$. The original description of $B(\infty)$ by Kashiwara was given in terms of complicated algebraic constructions. However, for specific algebras there often exists equivalent simple combinatorial realizations of $B(\infty)$. In this work, we study two such realizations for semi-simple Lie algebras of types B_n and C_n. The first realization of $B(\infty)$ is given in the language of the marginally large tableaux described by [5], and the second is in terms of the bracketing rules on Kostant partitions given in [10]. In the results of [3] and [11] these two realizations are studied for crystals of types A_n and D_n, respectively. In the type B_n and C_n situations considered here the description of the isomorphism is slightly more complicated. The complication arises because, as in the type D_n case, there is no longer a one-to-one correspondence between the tableau boxes and the positive roots. Further, the difference of number of paired and paired boxes in a row of a tableau affects how pairs of boxes and positive roots correspond. After giving a detailed construction and examples of each realization, an algorithmic description of the isomorphism is stated and a proof is given. Finally, following with the work of [11] we give a diagrammatic stack notation of the Kostant partitions for types B_n and C_n. The diagrams of the stack notation are suggestive of the action of the crystal operators on elements of the Kostant partition realization of $B(\infty)$.

CHAPTER II

BACKGROUND

In the first section we build up the underlying concepts necessary to give a definition of the crystal $B(\infty)$. Then, in the next section we provide the specialization of $B(\infty)$ to types B_n and C_n. In the last two sections of the background material we provide definitions of the type B_n and C_n crystal structures realized by marginally large tableaux and by Kostant partitions.

Unless otherwise noted, the exposition in this chapter follows [4].

II.1. Preliminaries

Definition II.1. Let I be a finite index set. Let $A = (a_{ij})_{i,j \in I}$ be a square matrix with $a_{ij} \in \mathbf{Z}$. The matrix A is a *generalized Cartan matrix* if

1. $a_{ii} = 2$ for all $i \in I$,
2. $a_{ij} \leq 0$ if $i \neq j$,
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

The Cartan matrix A is said to be *symmetrizable* if there exist a diagonal matrix with integer entries D such that DA is symmetric.

The matrix A is *indecomposable* if for every pair of nonempty disjoint subsets $I_1, I_2 \subset I$ with $I_1 \sqcup I_2 = I$, there exists a $i \in I_1$ and a $j \in I_2$ such that $a_{ij} \neq 0$.

From now on, all generalized Cartan matrices are assumed to be both symmetrizable and indecomposable.

With any generalized Cartan matrix we will be able to make a structure known as Kac-Moody algebra, which we define later. In order to build up a Kac-Moody algebra, we must associate to the generalized Cartan matrix some additional information. This required data is referred to as the *Cartan datum* which we describe in the following definition.

Definition II.2. Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix. The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ forms a *Cartan datum* for A with the following definitions.

1. The *dual weight lattice* $P^\vee$ is a free abelian group of rank $2|I| - \operatorname{rank} A$ with $\mathbf{Z}$-basis $\{\alpha_i^\vee \mid i \in I\} \cup \{d_s \mid s = 1, \dots, |I| - \operatorname{rank} A\}$.

2. The *weight lattice* P is defined as

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbf{Z}\},$$

where $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$ is the *Cartan subalgebra* and $\mathfrak{h}^*$ is its dual.

3. The *simple coroots* $\Pi^\vee$ are defined as

$$\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}.$$

4. Define $\langle\ ,\ \rangle \colon P^\vee \times P \to \mathbf{Z}$ to be the canonical pairing; that is, $\langle h, \lambda \rangle = \lambda(h)$ for all $h \in P^\vee$ and $\lambda \in P$. The *simple roots* Π are a linearly independent subset of $\mathfrak{h}^*$, $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$, where $\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$ and $\langle d_s, \alpha_j \rangle = 0$ or $\langle d_s, \alpha_j \rangle = 1$.

Following the notation of Definition II.2 the group $Q = \bigoplus_{i \in I} \mathbf{Z}\alpha_i$ is called the *root lattice* and $Q_+ = \sum_{i \in I} \mathbf{Z}_{\geq 0}\alpha_i$ is called the *positive root lattice*. We also define the *fundamental weights* ω_i for $i \in I$ to be the linear functionals on $\mathfrak{h}$ given by

$$\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij} \text{ and } \langle d_s, \omega_j \rangle = 0.$$

To allow the construction of a Kac-Moody algebra from the the Cartan datum we consider the group of simple reflections on the Cartan subalgebra $\mathfrak{h}^*$, known as the *Weyl group*.

Definition II.3. Consider the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ with Cartan subalgebra $\mathfrak{h}^*$. For each $i \in I$, define the *simple reflection* s_i on $\mathfrak{h}^*$ by

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i.$$

Let $W \subset \mathrm{GL}(\mathfrak{h}^*)$ be the subgroup generated by all simple reflections $\{s_i \mid i \in I\}$. Then W is called the *Weyl group* for the associated Cartan datum.

For any $w \in W$, $w = s_{i_1} s_{i_2} \cdots s_{i_t}$ is called a *reduced expression* (with length t) if t is the minimal number of simple reflections necessary to express w.

Now with the definition of the Cartan datum we have the necessary information to define the associated *Kac-Moody Algebra*. But first, we recall the definition of a Lie algebra. A *Lie algebra* L is a $\mathbf{C}$-vector space equipped with a bilinear map $[\ ,\ ]\colon L \times L \to L$, called the bracket, such that

1. $[x,x] = 0$ for all $x \in L$, and
2. $\big[x,[y,z]\big] + \big[y,[z,x]\big] + \big[z,[x,y]\big] = 0$.

The first equation describes the condition of *alternativity* and the latter equation is called the *Jacobi identity*. For each element of a Lie algebra there exists an endomorphism known as the *adjoint* given in terms of the bracket. That is, for every $x \in L$, there is a linear map $\mathrm{ad}(x)\colon L \to L$ defined by $\mathrm{ad}(x)(y) = [x,y]$ for all $y \in L$.

Definition II.4. Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum with generalized Cartan matrix A and index set I. The *Kac-Moody algebra*, $\mathfrak{g}$, associated with $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by the elements E_i, F_i, for $i \in I$, and $h \in P^\vee$ where,

1. $[h,h'] = 0$ for $h, h' \in P^\vee$,
2. $[E_i, F_j] = \delta_{ij} \alpha_i^\vee$,
3. $[h, E_i] = \alpha_i(h) E_i$ for $h \in P^\vee$,
4. $[h, F_i] = -\alpha_i(h) F_i$ for $h \in P^\vee$,
5. $(\mathrm{ad}\, E_i)^{1-a_{ij}} E_j = 0$ for $i \neq j$,
6. $(\mathrm{ad}\, F_i)^{1-a_{ij}} F_j = 0$ for $i \neq j$.

Further, define $\mathfrak{g}_+$ (resp. $\mathfrak{g}_-$) as the subalgebra of $\mathfrak{g}$ generated by E_i (resp. F_i).

Let q be an indeterminate. Before defining the quantum group, we first need the notion of q-integers, q-factorials, and q-binomials. For any $n \in \mathbf{Z}$, we define a q-integer as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbf{Z}[q, q^{-1}],$$

and a q-factorial as $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. We set $[0]_q! = 1$ by convention. Then, for $m,n \in \mathbf{N}$, define a q-binomial as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!} \in \mathbf{Z}[q,q^{-1}].$$

Definition II.5. Let $\mathfrak{g}$ be a Kac-Moody algebra associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ and finite index set I. The *quantum group*, $U_q(\mathfrak{g})$ is the associative algebra over $\mathbf{C}(q)$ with 1 generated by the elements E_i, F_i $(i \in I)$ and q^h $(h \in P^\vee)$ with the following relations.

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$
2. $q^h E_i q^{-h} = q^{\alpha_i(h)} E_i$ for $h \in P^\vee$
3. $q^h F_i q^{-h} = q^{-\alpha_i(h)} E_i$ for $h \in P^\vee$
4. $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0$ for $i \neq j$
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0$ for $i \neq j$

Here $q_i = q^{\gamma_i}$ and $K_i = q_i^{\alpha_i^\vee}$, where $D = \mathrm{diag}(\gamma_i : i \in I)$ is such that DA is a symmetric matrix.

Let $U_q^+(\mathfrak{g})$, $U_q^0(\mathfrak{g})$, and $U_q^-(\mathfrak{g})$ be the subalgebras of $U_q(\mathfrak{g})$ generated by E_i, $\mathfrak{h}$, and F_i respectively. Note that by the Poincaré-Birkhoff-Witt theorem it can be shown that the Kac-Moody algebra $\mathfrak{g}$ and its corresponding quantum group $U_q(\mathfrak{g})$ all have the same representation theory. Some of the details of the representation theory of quantum groups is shown in the following proposition from [4].

Proposition II.6. *Let $\mathfrak{g}$ be a Kac-Moody algebra with quantum group $U_q(\mathfrak{g})$. Then we have*

1. *the triangular decomposition given by $U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g})$ as $\mathbf{C}(q)$-vector spaces.*
2. *the root space decomposition given by $U_q(\mathfrak{g}) = \bigoplus_{\beta \in Q} (U_q(\mathfrak{g}))_\beta$, where*

$$(U_q(\mathfrak{g}))_\beta = \{x \in U_q(\mathfrak{g}) : q^h x q^{-h} = q^{\beta(h)} x \textit{ for all } h \in P^\vee\}.$$

We now provide the following general notion of crystals associated with a quantum group.

Definition II.7. Let $U_q(\mathfrak{g})$ be the quantum group associated with the Cartan datum $(A,\Pi,\Pi^\vee,P,P^\vee)$ and finite index set I. An *abstract $U_q(\mathfrak{g})$-crystal*, is a set B together with the maps $\mathrm{wt}\colon B \to P$, $e_i, f_i\colon B \to B \cup \{0\}$, and $\varepsilon_i, \varphi_i\colon B \to \mathbf{Z} \cup \{-\infty\}$, for all $i \in I$, satisfying the following properties:

1. $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle$ for all $i \in I$,
2. $\mathrm{wt}(e_i b) = \mathrm{wt}(b) + \alpha_i$ if $e_i b \in B$,
3. $\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i$ if $f_i b \in B$,
4. $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$, $\varphi_i(e_i b) = \varphi_i(b) + 1$ if $e_i b \in B$,
5. $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$, $\varphi_i(f_i b) = \varphi_i(b) - 1$ if $f_i b \in B$,
6. $f_i b = b'$ if and only if $b = e_i b'$ for $b, b' \in B, i \in I$,
7. If $\varphi_i(b) = -\infty$ for $b \in B$, then $e_i b = f_i b = 0$.

Definition II.8. Let B_1, B_2 be abstract $U_q(\mathfrak{g})$-crystals. A *crystal morphism* $\Psi\colon B_1 \to B_2$ is a map $\Psi\colon B_1 \cup \{0\} \to B_2 \cup \{0\}$ such that:

1. $\Psi(0) = 0$,
2. if $b \in B_1$ and $\Psi(b) \in B_2$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I$, and
3. if $b, b' \in B_1$, $\Psi(b)$, $\Psi(b') \in B_2$, and $f_i b = b'$, then $f_i \Psi(b) = \Psi(b')$ and $\Psi(b) = e_i \Psi(b')$ for all $i \in I$.

If the crystal morphism Ψ is a bijection between $B_1 \cup \{0\}$ and $B_2 \cup \{0\}$, then Ψ is a *crystal isomorphism*. If Ψ is a crystal isomorphism then the two abstract $U_q(\mathfrak{g})$-crystals, B_1 and B_2, are said to be *isomorphic*.

Finally we consider the structure obtained when taking the $q \to 0$ limit for a $U_q(\mathfrak{g})$-crystal. We refer this structure as the infinity crystal $B(\infty)$. The $B(\infty)$ crystal encodes information about the highest weight representations of the negative half of a quantum group. Since we will be working with $B(\infty)$ crystals of specific types, we now specialize these general definitions for the cases we deal with in the next section.

Table 1. Positive Roots of Type B_n, Expressed Both as a Linear Combination of Simple Roots and in the Canonical Realization following [2]

$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \le i \le k \le n$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_n,$	$1 \le i < k \le n$
$\beta_{i,k} = \varepsilon_i - \varepsilon_{k+1},$	$1 \le i \le k \le n-1$
$\beta_{i,n} = \varepsilon_i,$	$1 \le i \le n$
$\gamma_{i,k} = \varepsilon_i + \varepsilon_k,$	$1 \le i < k \le n$

Table 2. Positive Roots of Type C_n, Expressed Both as a Linear Combination of Simple Roots and in the Canonical Realization following [2]

$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \le i \le k < n$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_k,$	$1 \le i \le k \le n$
$\beta_{i,k} = \varepsilon_i - \varepsilon_{k+1},$	$1 \le i \le k < n$
$\gamma_{i,k} = \varepsilon_i + \varepsilon_k,$	$1 \le i \le k \le n$

II.2. Crystals of Type B_n and C_n

Set $I = \{1, 2, \ldots, n\}$. Let $\mathfrak{g}$ be the Lie algebra of type B_n or C_n with Cartan matrix $A = (a_{ij})$, as given respectively by

$$B_n : A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}, \qquad C_n : A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Let $\{\alpha_1, \ldots, \alpha_n\}$ be the simple roots and $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ the simple coroots, related by the inner product $\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$. Define the fundamental weights $\{\omega_1, \ldots, \omega_n\}$ by $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. Then the weight lattice is $P = \mathbf{Z}\omega_1 \oplus \cdots \oplus \mathbf{Z}\omega_n$ and the coweight lattice is $P^\vee = \mathbf{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbf{Z}\alpha_n^\vee$. The Cartan subalgebra $\mathfrak{h}$ is given by $\mathbf{C} \otimes_{\mathbf{Z}} P^\vee$. Also, let Φ denote the roots associated to $\mathfrak{g}$, with the set of positive roots denoted Φ^+.

The list of positive roots for crystals of types B_n and C_n are given in Tables 1 and 2, respectively. As before, let W be the Weyl group, generated by simple reflections $\{s_i \mid i = 1,2,\dots,n\}$. In these types, there exists a unique longest element of W, which is denoted by w_0.

Let e_i, f_i be the Kashiwara operators on $U_q^-(\mathfrak{g})$ defined in [7]. Let $\mathcal{A} \subset \mathbf{Q}(q)$ be the subring of functions regular at $q = 0$ and define $L(\infty)$ to be the $\mathcal{A}$-lattice spanned by

$$S = \{f_{i_1} f_{i_2} \cdots f_{i_t} \cdot 1 \in U_q^-(\mathfrak{g}) \mid t \geq 0,\ i_k \in I\}.$$

Now using the Cartan datum defined above we define $B(\infty)$ for crystals of type B_n and C_n.

Theorem/Definition II.9 ([7])**.**

1. Let $\pi\colon L(\infty) \to L(\infty)/qL(\infty)$ be the natural projection and set $B(\infty) = \pi(S)$. Then $B(\infty)$ is a $\mathbf{Q}$-basis of $L(\infty)/qL(\infty)$.

2. For each $i \in I$ the operators e_i and f_i act on $L(\infty)/qL(\infty)$. Moreover, $e_i\big(B(\infty)\big) = B(\infty) \sqcup \{0\}$ and $f_i\big(B(\infty)\big) \subset B(\infty)$.

For $i \in I$ and $b \in B(\infty)$, define

$$\varepsilon_i(b) = \max\{k \in \mathbf{Z}_{\geq 0} \mid e_i^k b \neq 0\}.$$

Consider the weight map $\mathrm{wt}\colon B(\infty) \to P$ defined by

$$\mathrm{wt}(f_{i_1} f_{i_2} \cdots f_{i_t} \cdot 1) = -\alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_t}.$$

II.3. Marginally Large Tableaux of Types B_n and C_n

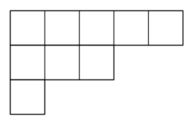
We now define the combinatorial object known as a *marginally large tableau*. First, we build up the definition of general Young diagrams and tableaux which we then specialize to *marginally large tableaux*.

An *integer partition* of a positive integer n is a way to write n as a sum of positive integers, known as *parts*. Integer partitions are considered equivalent under reordering of the summands. A Young diagram is a commonly used tool to provide a graphical depiction of an integer partition.

Definition II.10. A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row, reading rows from top to bottom.

So, the Young diagram for a partition of a positive integer n will consist of n total boxes, with each part corresponding to a row with length equal to the part.

Example II.11. Let $n = 9$ and consider the partition $9 = 5+3+1$. Then the associated Young diagram is



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Definition II.12. A *Young tableau* is a Young diagram with entries placed in each box from some ordered set $S = \{s_1 \prec s_2 \prec \cdots \prec s_k\}$. A *semistandard Young tableau* is a Young tableau obtained from filling a Young diagram with entries from S such that the following conditions are satisfied.

1. The entries in each row are weakly increasing from left-to-right.
2. The entries in each column are strictly increasing from top-to-bottom.

Example II.13. Continuing the last example, the filling

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

is a semistandard Young tableaux, but the filling

$$\begin{array}{|c|c|c|c|c|}\hline 2 & 3 & 5 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array}$$

is not.

Now, we describe an explicit realization of $B(\infty)$ for Lie algebras of types B_n and C_n by way of specialized semi-standard Young tableaux with a particular alphabet. We call these special

Young tableaux *marginally large tableaux*, and equip these diagrams with a mechanism for the action of the Kashiwara operators.

Definition II.14. Let the alphabets $J(B_n)$ and $J(C_n)$ be the totally ordered sets with order operation $\prec$ given below.

$$J(B_n) := \{1 \prec \cdots \prec n-1 \prec n \prec 0 \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}\}, \text{ and}$$
$$J(C_n) := \{1 \prec \cdots \prec n-1 \prec n \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}\}.$$

Define the set of marginally large tableaux, $\mathcal{T}(\infty)$, to be the set of semi-standard Young tableau with entries in alphabets $J(B_n)$ or $J(C_n)$, respectively, which satisfies the following conditions for all $T \in \mathcal{T}(\infty)$,

1. The number of $\boxed{i}$ in the i-th row of T is exactly one more than the total number of boxes in the $(i+1)$-th row.
2. Entries weakly increase along rows.
3. All entries in the i-th row are $\preceq \bar{\imath}$.
4. If T is of type B_n, the $\boxed{0}$ does not appear more than once per row.

Note that as a result of this definition, the leftmost column of T will contai the elements $\boxed{1}, \boxed{2}, \ldots, \boxed{n-1}, \boxed{n}$ in increasing order. Also, all of the $\boxed{i}$ in the i-th row of T are known as *shaded boxes* and shown as gray when drawing T in order to highlight the marginally large structure. In the subsequent examples, we depict the general structure for the two types of marginally large tableaux, where the partitions have three parts. The number of shaded boxes in each row, is one more than the total number of boxes in the next row. The notation $\boxed{i \cdots i}$ in the table below indicates any number of $\boxed{i}$ (possibly zero) and the $\boxed{0}$ in each row may or may not be present.

Example II.15. In type B_3, the elements of $\mathcal{T}(\infty)$ all have the form

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1\cdots1 & 1 & 1\cdots1 & 1 & 1\cdots1 & 1\cdots1 & 1 & 2\cdots2 & 3\cdots3 & 0 & \overline{3}\cdots\overline{3} & \overline{2}\cdots\overline{2} & \overline{1}\cdots\overline{1} \\ \hline 2 & 2 & 2\cdots2 & 2 & 3\cdots3 & 0 & \overline{3}\cdots\overline{3} & \overline{2}\cdots\overline{2} \\ \cline{1-8} 3 & 0 & \overline{3}\cdots\overline{3} \\ \cline{1-3} \end{array}.$$

Example II.16. In type C_3, the elements of $\mathcal{T}(\infty)$ all have the form

$$T = \begin{array}{cccccccccccc} 1 & 1\cdots1 & 1 & 1\cdots1 & 1\cdots1 & 1\cdots1 & 1 & 2\cdots2 & 3\cdots3 & \overline{3}\cdots\overline{3} & \overline{2}\cdots\overline{2} & \overline{1}\cdots\overline{1} \\ 2 & 2\cdots2 & 2 & 3\cdots3 & \overline{3}\cdots\overline{3} & \overline{2}\cdots\overline{2} & & & & & & \\ 3 & \overline{3}\cdots\overline{3} & & & & & & & & & & \end{array}.$$

Shown in Figure 1 and Figure 2, we give what are known as the fundamental crystals for types B_n and C_n. The fundamental crystal, denoted by $B(\omega_1)$, is the crystal graph associated to the fundamental representation $V(\omega_1)$ of $U_q(\mathfrak{g})$ of highest weight ω_1. The crystal structure on $\mathcal{T}(\infty)$ is then determined by embedding $\mathcal{T}(\infty)$ into a tensor product $B(\omega_1)^{\otimes N}$, for a suitably large positive integer N. With the combinatorial construction of $\mathcal{T}(\infty)$ we will not require the details of irreducible highest weight representations or crystal tensor products. Here we will assume the existence of fundamental crystals without full definition and use it to describe the crystal structure on $\mathcal{T}(\infty)$. An interested reader should refer to [4] for the full definition of the fundamental crystal in terms of the tensor product terminology.

Since, by the crystal axioms, e_i and f_i will both adjust the weight of a tableau by one simple root, they will act by increasing or decreasing the entry of a single box by one in the order on $J(B_n)$ or $J(C_n)$. The fundamental crystal shows how e_i and f_i could modify a tableaux. The i-colored arrows between the boxes indicate that f_i will move the box on the left end of the arrow to the right end, and that e_i will move the box at the right end to the left. Our subsequent definitions of the weight function and the details of the action of the Kashiwara operators for $\mathcal{T}(\infty)$ were derived using this structure.

$$\boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}}$$

Figure 1. The Fundamental Crystal of Type B_n

$$\boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}}$$

Figure 2. The Fundamental Crystal of Type C_n

Each marginally large tableaux of type B_n or C_n corresponds to a sum of the simple roots of $\mathfrak{g}$. We call this sum the *weight* of the tableau, which by convention we take to be the negative of the sum of the simple roots. The weight of the tableau can be determined by considering the weight of each box as defined by the statement we give below.

Definition II.17. Let $T \in \mathcal{T}(\infty)$. First we define the weight of a specific box. All shaded boxes are defined to be of zero weight. Let $\boxed{k}_j$ be an unshaded box containing k in the j-th row of T. In type B_n, define the weight of each box as

$$\mathrm{wt}\left(\boxed{k}_j\right) = \begin{cases} -\beta_{j,k-1} & \text{if } k \text{ is an unbarred letter,} \\ -\beta_{j,n} & \text{if } k = 0, \\ -\gamma_{j,k} & \text{if } k \text{ is a barred letter but } k \neq \overline{j}, \\ -2\beta_{j,n} & \text{if } k = \overline{j}. \end{cases}$$

In type C_n, define the weight of each box as

$$\mathrm{wt}\left(\boxed{k}_j\right) = \begin{cases} -\beta_{j,k-1} & \text{if } k \text{ is an unbarred letter,} \\ -\gamma_{j,k} & \text{if } k \text{ is a barred letter.} \end{cases}$$

In both cases, define the weight of T as the sum over all unshaded boxes,

$$\mathrm{wt}(T) = \sum_{\boxed{k}_j \in T} \mathrm{wt}\left(\boxed{k}_j\right).$$

Note that we define the weight to be negative in line with the crystal axioms in Definition II.7 which state that each Kashwara lowering operator will lower the weight of a $T \in \mathcal{T}(\infty)$ by one simple root.

As shown in Examples II.15 and II.16 we typically shade the i-boxes in row i, as these do not contribute to the weight of the tableau. In particular, the unique element of weight zero for both type B_3 and C_3 is

$$T_\infty = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}\,.$$

We now demonstrate the calculation of the weight for a small tableau.

Example II.18. Let $T \in \mathcal{T}(\infty)$ be of type B_3 with

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & \bar{1} \\ \hline 2 & 2 & 2 & 2 & \bar{3} \\ \cline{1-5} 3 & 0 & \bar{3} \\ \cline{1-3} \end{array}$$

Then, following Definition II.17,

$$\begin{aligned}
\mathrm{wt}\left(\boxed{2}_1\right) &= -\beta_{1,1} = -\alpha_1, \\
\mathrm{wt}\left(\boxed{\bar{1}}_1\right) &= -2\beta_{1,3} = -2\alpha_1 - 2\alpha_2 - 2\alpha_3, \\
\mathrm{wt}\left(\boxed{\bar{3}}_2\right) &= -\gamma_{2,3} = -\alpha_2 - 2\alpha_3, \\
\mathrm{wt}\left(\boxed{0}_3\right) &= -\beta_{3,3} = -\alpha_3, \\
\mathrm{wt}\left(\boxed{\bar{3}}_3\right) &= -2\beta_{3,3} = -2\alpha_3.
\end{aligned}$$

Hence

$$\mathrm{wt}(T) = -3\alpha_1 - 3\alpha_2 - 7\alpha_3.$$

Note that this weight does not uniquely define the tableau. Consider $T' \in \mathcal{T}(\infty)$, with

$$T' = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & \bar{3} & \bar{2} \\ \cline{1-6} 3 & 0 & \bar{3} \\ \cline{1-3} \end{array}\,.$$

Then $\mathrm{wt}(T) = \mathrm{wt}(T')$, but $T \neq T'$. The interesting thing to observe from this example, is that $T \neq T'$ with $\mathrm{wt}(T) = \mathrm{wt}(T')$. This implies that the weight must have multiple decompositions into the positive roots of Table 1. Determining the nature of these decompositions is the key to understanding the isomorphism proposed later in this paper. That is, we will see that stating the existence of an isomorphism between the marginally large tableaux and the Kostant partition realizations of the crystal $B(\infty)$ is equivalent to showing that each element of $\mathcal{T}(\infty)$ has a unique decomposition into positive roots which respects the crystal structure.

From the weight of a marginally large tableau we can also determine the *depth* of the tableau within the crystal $\mathcal{T}(\infty)$. By depth we mean the length of any path from the unique highest weight element, T_∞, to the given tableau. The depth is sum of the coefficients of the simple roots in the weight of the tableau. This results because, by the crystal axioms, each Kashiwara lowering operator decreases the weight by exactly one simple root.

In order to define Kashiwara operators on $\mathcal{T}(\infty)$ we first consider a listing of the boxes of a $T \in \mathcal{T}(\infty)$ as a word on the entries of T.

Definition II.19. Let $T \in \mathcal{T}(\infty)$.

1. The *Far-Eastern reading* of T, denoted $\mathrm{read}_{\mathrm{FE}}(T)$, records the entries of the boxes in the columns of T from top to bottom and proceeding from right to left.
2. The *Middle-Easten reading* of T, denoted $\mathrm{read}_{\mathrm{ME}}(T)$, records the entries of the boxes in the rows of T from right to left and proceeding from top to bottom.

For either reading on $T \in \mathcal{T}(\infty)$ we can define the action of the Kashiwara operators e_i and f_i by way of a bracketing sequence constructed from that reading. This bracketing sequence, and corresponding operators, are obtained by converting the letters of this word into a sequence of left and right brackets, sequentially canceling pairs, and considering the letters which contribute the leftmost left and rightmost right brackets after cancellation. The result of this conversion and cancellation is known as the *i-signature* of a tableau T. In the following definition, we give this bracketing sequence and *i-signature* in terms of either reading.

Definition II.20. Let $T \in \mathcal{T}(\infty)$ of type B_n or C_n, and set $\mathrm{read}(T) = \mathrm{read}_{\mathrm{ME}}(T)$ or $\mathrm{read}_{\mathrm{FE}}(T)$. Consider the fundamental crystals depicted in Figures 1 and 2. For each $i \in \{1,2,\ldots,n\}$, the bracketing sequence $\mathrm{br}_i(T)$ is obtained by replacing each letter in $\mathrm{read}(T)$, with $)^p(^q$, where p is number of consecutive i-arrows entering and q is the number of consecutive i-arrows leaving the corresponding box in the fundamental crystal. After determining $\mathrm{br}_i(T)$, sequentially cancel all ()-pairs to obtain a sequence of the form $)\cdots)(\cdots($. Denote by $\mathrm{br}_i^c(T)$ the sequence of brackets which is obtained after cancellation. We refer to sequence of brackets $\mathrm{br}_i^c(T)$ as the *i-signature*.

Example II.21. Let $T \in \mathcal{T}(\infty)$ for $\mathfrak{g}$ of type B_3 where

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&1&1&2&3&0&\overline{3}&\overline{2}\\ \hline 2&2&2&2&2&3&0&\overline{3}&\overline{3}&\overline{2}\\ \cline{1-10} 3&0&\overline{3}&\overline{3}\\ \cline{1-4}\end{array} .$$

Following Definition II.20 we obtain these bracketing sequences and i-signatures for T.

$$\begin{aligned}
\mathrm{read}_{\mathrm{ME}}(T) &= \overline{2}\ \overline{3}\ 0\ 3\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \overline{2}\ \overline{3}\ \overline{3}\ 0\ 3\ 2\ 2\ 2\ 2\ 2\ \overline{3}\ \overline{3}\ 0\ 3\ ,\\
\mathrm{br}_1(T) &= (\quad)\ (\ (\ (\ (\ (\ (\ (\ (\ (\ (\ (\ (\quad)\ )\ )\ )\ )\ ,\\
\mathrm{br}_1^c(T) &= (\ (\ (\ (\ (\ (\ (\ ,\\
\mathrm{br}_2(T) &=)\ (\quad (\ )\quad)\ (\ (\quad)\ (\ (\ (\ (\ (\ (\ (\quad)\ ,\\
\mathrm{br}_2^c(T) &=)\quad (\quad (\ (\ (\ (\ (\ (\ ,\\
\mathrm{br}_3(T) &=))\ )(\ ((\quad))\ ))\ )(\ ((\quad))\ ))\ )(\ ((\ .\\
\mathrm{br}_3^c(T) &=))\ )\quad)\ )\quad)\ )(\ ((\ .
\end{aligned}$$

These i-signatures on elements of $\mathcal{T}(\infty)$ encode the information necessary to define the Kashiwara operators. The action of the operators follows the description given next.

Definition II.22. Let $T \in \mathcal{T}(\infty)$ and $i \in I$.

1. Let $\boxed{r}$ be the box in T corresponding to the rightmost ')' in $\mathrm{br}_i^c(T)$. Then e_iT is the tableau obtained from T by replacing the r in $\boxed{r}$ with the predecessor in the alphabet of $\mathcal{T}(\infty)$. The result will always be marginally large unless $\boxed{r}$ occurs in row i and r is the successor for i in the alphabet of $\mathcal{T}(\infty)$. In this case, e_iT is obtained by deletion of the column containing $\boxed{r}$.

2. Let $\boxed{\ell}$ be the box in T corresponding to leftmost '(' in $\mathrm{br}_i^c(T)$. Then f_iT is the tableau obtained from T by replacing the ℓ in $\boxed{\ell}$ with the successor of ℓ in the alphabet of

$\mathcal{T}(\infty)$. The result will always be marginally large unless $\boxed{\ell}$ occurs in row i and $\ell = i$, in which case f_iT is obtained by replacing the ℓ in $\boxed{\ell}$ with the next letter in the alphabet, and then inserting a column with the elements $\boxed{1}, \boxed{2}, \ldots, \boxed{i}$ directly to the left of $\boxed{\ell}$.

Proposition II.23 ([5])**.** *Using* $\mathrm{read}_{\mathrm{FE}}(T)$ *for the operations defined above,* $\mathcal{T}(\infty)$ *is isomorphic to* $B(\infty)$ *as a crystal.* ∎

It is important here to note that Proposition II.23 from [5] uses the *Far-Eastern reading* to define e_i, f_i, but we prefer to use the *Middle-Eastern reading* to simplify later arguments. The two reading words are certainly not the same, so we must provide justification for our alternative use of the Middle-Eastern reading in the determination of the i-signature for a marginally large tableau. In the work of Hong and Lee [5] the realization of $B(\infty)$ by marginally tableaux is obtained by way of the Far-Eastern reading. However, we claim here the Middle-Eastern and the Far-Eastern readings yield equivalent crystal structures on marginally large tableaux. By equivalent, we mean that the words may differ, but that the leftmost and rightmost brackets in the canceled bracket sequence correspond to the same entries in the tableaux. By this equivalence the actions of the Kashiwara operators can be defined in the same way as in [5], and the resulting construction is also isomorphic to $B(\infty)$ for semi-simple Lie algebras of types B_n or C_n. This claim is stated and proved in the subsequent proposition. Immediately following the proof we give two examples to demonstrate how to calculate the described procedure for the action of Kashiwara operators, e_i and f_i, on marginally large tableaux of type B_n or C_n.

Proposition II.24. *Let* $\mathcal{T}(\infty)$ *be the marginally large tableau realization of* $B(\infty)$ *for a semi-simple Lie algebra of type* B_n *or* C_n*. Then the definition of the Kashiwara operators,* e_i *and* f_i*, on* $\mathcal{T}(\infty)$ *is equivalent when using either the Far-Eastern reading or the Middle-Eastern reading.*

Proof. First, we note that the calculation of the operators, e_i and f_i, depends only on the determination of the i-signature, br_i. So we need only show that the i-signatures are equivalent under the two readings. This will be done by showing the leftmost '(' and the rightmost ')' in br_i^c correspond to the same boxes in a given tableau.

Also, consider that for a given tableau, T, the i-signature is determined by the position of $\boxed{i}$, $\boxed{i+1}$, $\boxed{\bar{i}}$, $\boxed{\overline{i+1}}$ (and possibly $\boxed{0}$ if $i = n$ and T is of type B_n) boxes in T. Because of the marginally large condition imposed on T, these boxes only occur within the first $i+1$ rows. Thus all brackets used in calculation of the i-signature will be determined from relevant boxes in rows $1, \dots, i+1$. Computation of the i-signature can be achieved using either $\mathrm{read}_{\mathrm{FE}}(T)$ or $\mathrm{read}_{\mathrm{ME}}(T)$.

Let $T \in \mathcal{T}(\infty)$ of type B_n or C_n. To simplify the argument, we define $\ell_{i,j}$ to be the number of $\boxed{j}$ boxes in row i of T. Divide the bracketing sequence from each reading into two parts, a *suffix* and a *prefix*. For the Middle-Eastern reading define the *prefix* as the part of $\mathrm{read}_{\mathrm{ME}}(T)$ determined by reading up until the rightmost $\boxed{i}$ in row i of T, and define the *suffix* as the remaining part of the word. For the Far-Eastern reading define the *prefix* as the part of $\mathrm{read}_{\mathrm{FE}}(T)$ found by reading up until the start of the column containing the rightmost $\boxed{i}$ in the i-th row of T, and the *suffix* is the remaining sequence. That is the suffix for the $\mathrm{read}_{\mathrm{FE}}(T)$ will consist of the first $\ell_{i,i}$ columns of T.

We claim that the prefix for the Middle-Eastern and Far-Eastern readings are identical because for this portion the words obtained will the same under either reading. Consider row j with $j \leq i$. Due to the marginally large condition on T all relevant boxes in row j occur southwest of a $\boxed{j-1}$ in row $j-1$. Also, all relevant boxes in row $j-1$ occur east of $\boxed{j-1}$ boxes. Thus no relevant box in row j occurs in the same column as relevant boxes from row $j-1$. So when reading the Far-Eastern prefix each column can contribute at most one box, and they fall in the same order as when reading the Middle-Eastern prefix. Therefore the prefix sequences are identical for the Middle-Eastern and Far-Eastern readings.

It now remains to show that the suffixes under either reading determine equivalent bracketing sequences. For the suffix of the sequences we have two distinct cases.

Case 1: $i < n$. Fix $T \in \mathcal{T}(\infty)$. Consider the bracketing sequences obtained by reading T by both Middle-Eastern and Far-Eastern readings. Computing the bracketing sequence for each suffix yields the following,

$$\text{Middle-Eastern: } \cdots (^{\ell_{i,i}} (^{\ell_{i+1,\overline{i+1}}})^{\ell_{i+1,i+1}},$$
$$\text{Far-Eastern: } \cdots (^{\ell_{i,i}+\ell_{i+1,\overline{i+1}}-\ell_{i+1,i+1}} \underbrace{()\cdots()}_{\ell_{i+1,i+1}}.$$

Since $\ell_{i,i} > \ell_{i+1,i+1}$, after cancellation the terminal sequences contain only '('. Furthermore, in both the Middle-Eastern and the Far-Eastern reading, the leftmost '(' in the suffix will correspond to the first box that is read, namely the rightmost $\boxed{i}$ in row i. Recall that the actions of e_i and f_i only depend on the boxes associated to the rightmost ')' and the leftmost '(', respectively. Since the suffixes contains no '(', and the prefixes are identical, the leftmost '(' corresponds to the same box under either reading.

Case 2: $i = n$. Again we note that the prefixes are identical regardless of reading and show the suffixes form equivalent bracketing sequences. If $i = n$, the suffix consists of the bottom row read right-to-left starting at the $\ell_{n,n}$ position for the Middle-Eastern reading, and of the first $\ell_{n,n}$ columns read right-to-left for the Far-Eastern reading. For each of the first $\ell_{n,n}$ columns the only relevant box is the box that is also in the last row of the column. Thus reading the suffix by the Middle-Eastern convention will produce the identical bracketing sequence to reading the suffix by the Far-Eastern convention. Therefore the leftmost '(' and the rightmost ')' occur in the same position for each reading.

Since the prefixes are identical and the suffixes are equivalent, $\mathrm{br}_i^c(T)$ determines the same actions for e_i and f_i when found using either $\mathrm{read}_{\mathrm{ME}}(T)$ or $\mathrm{read}_{\mathrm{FE}}(T)$. ■

Corollary II.25. *Using* $\mathrm{read}_{\mathrm{ME}}(T)$ *for the operations defined in Definition II.22,* $\mathcal{T}(\infty)$ *is isomorphic to* $B(\infty)$ *as a crystal.* ■

From here on, we set $\mathrm{read}(T) = \mathrm{read}_{\mathrm{ME}}(T)$. With this assertion, note that $\mathrm{br}_i(T)$ factors as

$$\mathrm{br}_i(R_1)\mathrm{br}_i(R_2)\cdots\mathrm{br}_i(R_n),$$

where R_j is the j-th row of T, counting from the top.

Example II.26. Let $T \in \mathcal{T}(\infty)$ for $\mathfrak{g}$ of type B_3 where

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&2&0&\bar{3}&\bar{2}&\bar{1}&\bar{1}\\\hline 2&2&2&2&3&0&\bar{2}&\bar{2}\\\cline{1-8} 3&\bar{3}&\bar{3}\\\cline{1-3}\end{array}\,.$$

To calculate e_3 and f_3, we consider the relevant section of the fundamental crystal:

$$\boxed{3} \xrightarrow{3} \boxed{0} \xrightarrow{3} \boxed{\bar{3}}\,.$$

Thus each $\boxed{3}$ will contribute '((', the $\boxed{0}$ will give ')(', each $\boxed{\bar{3}}$ will give '))', and all other letters will contribute nothing. Applying Definition II.22, the reading word and bracketing sequence are

$$\begin{aligned}
\mathrm{read}(T) &= \bar{1}\ \bar{1}\ \bar{2}\ \bar{3}\ \ 0\ \ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \bar{2}\ \bar{2}\ \ 0\ \ 3\ 2\ 2\ 2\ 2\ \bar{3}\ \ \bar{3}\ \ 3\ ,\\
\mathrm{br}_3(T) &= \qquad\quad))\ )(\qquad\qquad\qquad\qquad\quad)(\ ((\qquad\quad))\ ))\ ((\ ,\\
\mathrm{br}_3^c(T) &= \qquad\quad))\ \ ) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad)\ \ ((\ .
\end{aligned}$$

The rightmost ')' in $\mathrm{br}_3^c(T)$ is the one shown in blue, so e_3 changes the corresponding $\boxed{\bar{3}}$ in the third row to a $\boxed{0}$. Hence,

$$e_3T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&2&0&\bar{3}&\bar{2}&\bar{1}&\bar{1}\\\hline 2&2&2&2&3&0&\bar{2}&\bar{2}\\\cline{1-8} 3&0&\bar{3}\\\cline{1-3}\end{array}\,.$$

The leftmost '(' in $\mathrm{br}_3^c(T)$ is the one shown in blue, so f_3 changes the corresponding $\boxed{3}$ in the third row to a $\boxed{0}$. To maintain the marginally large condition, we insert a column to the left of the new $\boxed{0}$. This gives,

$$f_3T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&1&2&0&\bar{3}&\bar{2}&\bar{1}&\bar{1}\\\hline 2&2&2&2&2&3&0&\bar{2}&\bar{2}\\\cline{1-9} 3&0&\bar{3}&\bar{3}\\\cline{1-4}\end{array}\,.$$

Example II.27. Let $T \in \mathcal{T}(\infty)$ for $\mathfrak{g}$ of type C_3 where

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{2} & \overline{1} \\
\hline
2 & 2 & 2 & 2 & 3 & \overline{3} & \overline{3} & \overline{1} \\
\cline{1-8}
3 & \overline{3} & \overline{3} \\
\cline{1-3}
\end{array} .$$

To calculate e_3 and f_3, we consider the relevant section of the fundamental crystal:

$$\boxed{3} \xrightarrow{\;3\;} \boxed{\overline{3}} .$$

Thus each $\boxed{3}$ will contribute '(', each $\boxed{\overline{3}}$ will give ')', and all other letters will contribute nothing. Applying Definition II.22, the reading word and bracketing sequence are

$$\begin{aligned}
\mathrm{read}(T) &= \overline{1}\ \overline{2}\ \overline{3}\ 3\ 3\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \overline{1}\ \overline{3}\ \overline{3}\ 3\ 2\ 2\ 2\ 2\ \overline{3}\ \overline{3}\ 3\ , \\
\mathrm{br}_3(T) &= \quad)\ (\ (\qquad\qquad\qquad\qquad)\ )\ (\qquad\quad)\ )\ (\ , \\
\mathrm{br}_3^c(T) &= \quad) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad)\ (\ .
\end{aligned}$$

Recall the action of e_3 is determined by of the entry of the box corresponding to the rightmost ')' in br_3^c. The rightmost ')' in $\mathrm{br}_3^c(T)$ is the one shown in blue, which comes from the left $\boxed{\overline{3}}$ in the third row. The letter $\boxed{\overline{3}}$ immediately follows the row number, 3, in the alphabet so the action of e_3 is deletion of the column containing this $\overline{3}$. The effect of this action is

$$e_3T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{2} & \overline{1} \\
\hline
2 & 2 & 2 & 3 & \overline{3} & \overline{3} & \overline{1} \\
\cline{1-7}
3 & \overline{3} \\
\cline{1-2}
\end{array} .$$

The action of f_3 is the promotion by the entry of the box corresponding to the leftmost '(' in br_3^c. In this case, the leftmost '(' in br_3^c is the shown in blue which is associated to the $\boxed{3}$ in the third row. Thus the action of f_3 promotes the $\boxed{3}$ in the third row to a $\boxed{\overline{3}}$, and since this $\boxed{3}$ is in the third row, we insert a column with entries $\{1,2,3\}$ immediately to the left of this box. The resulting tableau is as follows.

$$f_3T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & \overline{3} & \overline{2} & \overline{1} \\
\hline
2 & 2 & 2 & 2 & 2 & 3 & \overline{3} & \overline{3} & \overline{1} \\
\cline{1-9}
3 & \overline{3} & \overline{3} & \overline{3} \\
\cline{1-4}
\end{array}$$

II.4. Crystal Structure on Kostant Partitions

Here we review the crystal structure on Kostant partitions from [10]. As explained there, this is naturally identified with the crystal structure on PBW monomials from, for example, [1, 9] for the reduced expression

$$w_0 = (s_1 s_2 \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_1) \cdots (s_{n-2} s_{n-1} s_n s_{n-2}) s_{n-1} s_n.$$

Let $\mathcal{R}$ be the set of symbols $\{(\beta) : \beta \in \Phi^+\}$. Let $\mathrm{Kp}(\infty)$ be the free $\mathbf{Z}_{\geq 0}$-span of $\mathcal{R}$. This is the set of *Kostant partitions*. We denote elements of $\mathrm{Kp}(\infty)$ by $\boldsymbol{\alpha} = \sum_{(\beta)\in\mathcal{R}} c_\beta (\beta)$.

As we did for $\mathcal{T}(\infty)$, we can define *weight* for elements of $\mathrm{Kp}(\infty)$ in terms of the positive roots which compose the element. In the Kostant partition situation the definition of the weight function is much more simple than what we saw for $\mathcal{T}(\infty)$. This is because the elements of $\mathrm{Kp}(\infty)$ are expressed as sums of positive roots, and the positive roots themselves are already given as sums of simple roots.

Definition II.28. Let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$. So $\boldsymbol{\alpha}$ can be expressed as $\boldsymbol{\alpha} = \sum\limits_{(\beta)\in\mathcal{R}} c_\beta (\beta)$. The weight, $\mathrm{wt}(\boldsymbol{\alpha})$, is then defined as

$$\mathrm{wt}(\boldsymbol{\alpha}) = -\sum_{\beta\in\Phi^+} c_\beta \beta.$$

To facilitate the development of a definition of the action of crystal operators, e_i and f_i, on the Kostant partitions, we will need a bracketing sequence similar to that for $\mathcal{T}(\infty)$. Unlike the $\mathcal{T}(\infty)$ case, here we do not have a reading word so in order to describe the bracketing sequences we first define sequences on specific subsets of the positive roots. After defining these sequences of positive roots, we can define an *i-signature* on each Kostant partition $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$. We denote the bracketing sequence for this signature as $S_i(\boldsymbol{\alpha})$, which gives us the i-signature, $S_i^c(\boldsymbol{\alpha})$, after cancellation.

Definition II.29. For crystals of type B_n, consider the following sequences of positive roots depending on $i \in I$.

$$S_i(\boldsymbol{\alpha}) = \begin{cases} \underbrace{)\cdots)}_{c_{\beta_{1,i}}} \ \underbrace{(\cdots(}_{c_{\beta_{1,i-1}}} \ \underbrace{)\cdots)}_{c_{\gamma_{1,i}}} \ \underbrace{(\cdots(}_{c_{\gamma_{1,i+1}}} \ \cdots \ \underbrace{)\cdots)}_{c_{\beta_{i-1,i}}} \ \underbrace{(\cdots(}_{c_{\beta_{i-1,i-1}}} \ \underbrace{)\cdots)}_{c_{\gamma_{i-1,i}}} \ \underbrace{(\cdots(}_{c_{\gamma_{i-1,i+1}}} \ \underbrace{)\cdots)}_{c_{\beta_{i,i}}} & \text{if } i < n, \\ \underbrace{)\cdots)}_{c_{\beta_{1,n}}} \ \underbrace{(\cdots(}_{2c_{\beta_{1,n-1}}} \ \underbrace{)\cdots)}_{2c_{\gamma_{1,n}}} \ \underbrace{(\cdots(}_{c_{\beta_{1,n}}} \ \cdots \ \underbrace{)\cdots)}_{c_{\beta_{n-1,n}}} \ \underbrace{(\cdots(}_{2c_{\beta_{n-1,n-1}}} \ \underbrace{)\cdots)}_{2c_{\gamma_{n-1,n}}} \ \underbrace{(\cdots(}_{c_{\beta_{n-1,n}}} \ \underbrace{)\cdots)}_{c_{\beta_{n,n}}} & \text{if } i = n. \end{cases}$$

Figure 3. Signature Rules for Elements of Kp(∞) of Type B_n

1. For $1 \le i \le n-1$, define the sequence

$$\Phi_i^B = (\beta_{1,i}, \beta_{1,i-1}, \gamma_{1,i}, \gamma_{1,i+1}, \ldots, \beta_{i-1,i}, \beta_{i-1,i-1}, \gamma_{i-1,i}, \gamma_{i-1,i+1}, \beta_{i,i}).$$

2. For $i = n$, define the sequence

$$\Phi_n^B = (\beta_{1,n}, \beta_{1,n-1}, \gamma_{1,n}, \beta_{1,n}, \ldots, \beta_{n-1,n}, \beta_{n-1,n-1}, \gamma_{n-1,n}, \beta_{n-1,n}, \beta_{n,n}).$$

For crystals of types C_n consider the following sequences of positive roots depending on $i \in I$.

1. For $1 \le i \le n-1$, define

$$\Phi_i^C = (\beta_{1,i}, \beta_{1,i-1}, \gamma_{1,i}, \gamma_{1,i+1}, \ldots, \beta_{i-1,i}, \beta_{i-1,i-1}, \gamma_{i-1,i-1}, \gamma_{i-1,i+1}, \beta_{i,i}).$$

2. For $i = n$, define

$$\Phi_n^C = (\gamma_{1,1}, \beta_{1,n-1}, \gamma_{1,n}, \gamma_{1,1}, \ldots, \gamma_{n-1,n-1}, \beta_{n-1,n-1}, \gamma_{n-1,n}, \gamma_{n-1,n-1}, \gamma_{n,n}).$$

We now define *bracketing sequence*, $S_i(\boldsymbol{\alpha})$, separately for the cases of $i < n$, $i = n$ with α of type B_n, and $i = n$ with α of type C_n.

Let $\boldsymbol{\alpha} \in$ Kp(∞) of type B_n or C_n. We build up $S_i(\boldsymbol{\alpha})$ by replacing each positive root in the sequences Φ_i^B or Φ_i^C with left and right brackets according to result of adding or removing α_i from each root. Figures 3 and 4 show the bracketing sequences that result from the construction.

$$S_i(\boldsymbol{\alpha}) = \begin{cases} \underbrace{)\cdots)}_{c_{\beta_{1,i}}} \; \underbrace{(\cdots(}_{c_{\beta_{1,i-1}}} \; \underbrace{)\cdots)}_{c_{\gamma_{1,i}}} \; \underbrace{(\cdots(}_{c_{\gamma_{1,i+1}}} \; \cdots \; \underbrace{)\cdots)}_{c_{\beta_{i-1,i}}} \; \underbrace{(\cdots(}_{c_{\beta_{i-1,i-1}}} \; \underbrace{)\cdots)}_{c_{\gamma_{i-1,i}}} \; \underbrace{(\cdots(}_{c_{\gamma_{i-1,i+1}}} \; \underbrace{)\cdots)}_{c_{\beta_{i,i}}} & \text{if } i < n, \\ \underbrace{)\cdots)}_{c_{\gamma_{1,1}}} \; \underbrace{(\cdots(}_{c_{\beta_{1,n-1}}} \; \underbrace{)\cdots)}_{c_{\gamma_{1,n}}} \; \underbrace{(\cdots(}_{c_{\gamma_{1,1}}} \; \cdots \; \underbrace{)\cdots)}_{c_{\gamma_{n-1,n-1}}} \; \underbrace{(\cdots(}_{c_{\beta_{n-1,n-1}}} \; \underbrace{)\cdots)}_{c_{\gamma_{n-1,n}}} \; \underbrace{(\cdots(}_{c_{\gamma_{n-1,n-1}}} \; \underbrace{)\cdots)}_{c_{\gamma_{n,n}}} & \text{if } i = n. \end{cases}$$

Figure 4. Signature Rules For Elements of $\mathrm{Kp}(\infty)$ of Type C_n

In each case, successively cancel all ()-pairs in $S_i(\boldsymbol{\alpha})$ to obtain sequence of the form $)\cdots)(\cdots($. We call the i-signature, $S_i^c(\boldsymbol{\alpha})$, the sequence which is obtained after cancellation.

Now that we have described how to determine the i-signatures for elements of $\mathrm{Kp}(\infty)$, we provide a definition of the action of the crystal operators which will show $\mathrm{Kp}(\infty)$ to be isomorphic to $B(\infty)$.

Definition II.30. Let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$ with $\boldsymbol{\alpha} = \sum_{(\beta)\in\mathcal{R}} c_\beta(\beta) \in \mathrm{Kp}(\infty)$. We then define the action of the crystal operators on $\boldsymbol{\alpha}$ by the following processes.

If $\mathrm{Kp}(\infty)$ is of type B_n with $i \in I$ or of type C_n with $i \in I \setminus \{n\}$, the action of e_i and f_i is determined as follows. Let β be the root corresponding to the rightmost ')' in $S_i^c(\boldsymbol{\alpha})$. Define

$$e_i\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta) + (\beta - \alpha_i).$$

Note that if $\beta = \alpha_i$, we interpret (0) as the additive identity in $\mathrm{Kp}(\infty)$. Furthermore, if no such ')' exists, then $e_i\boldsymbol{\alpha}$ is undefined.

Let γ denote the root corresponding to the leftmost '(' in $S_i^c(\boldsymbol{\alpha})$. Define,

$$f_i\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma) + (\gamma + \alpha_i).$$

If no such '(' exists, set $f_i\boldsymbol{\alpha} = \boldsymbol{\alpha} + (\alpha_i)$.

The above holds for all cases except for crystals of type C_n and $i = n$, where we have the following description of e_n and f_n.

- Let $\beta \in \Phi_n^C$ be the root corresponding to the rightmost ‘)’ in $S_n^c(\boldsymbol{\alpha})$. Define $e_n\boldsymbol{\alpha}$ as follows, for $k \in \{1,\ldots,n-1\}$. If no such β exists, then $e_n\boldsymbol{\alpha}$ is not defined.

  1. If $\beta = \gamma_{k,n}$ and $c_{\gamma_{k,n}} = c_{\beta_{k,n-1}} + 1$, then
  $$e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta) + (\beta_{k,n-1}).$$
  2. If $\beta = \gamma_{k,n}$ and $c_{\gamma_{k,n}} > c_{\beta_{k,n-1}} + 1$, then
  $$e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - 2(\beta) + (\gamma_{k,k}).$$
  3. If $\beta = \gamma_{k,k}$, then
  $$e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta) + 2(\beta_{k,n-1}).$$
  4. If $\beta = \gamma_{n,n}$, then
  $$e_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta).$$

- Let $\gamma \in \Phi_n^C$ denote the root corresponding to the leftmost ‘(’ in $S_n^c(\boldsymbol{\alpha})$. Define $f_n\alpha$ as follows, for $k \in \{1,\ldots,n\}$. If no such γ exists, proceed to (4).

  1. If $\gamma = \beta_{k,n-1}$ and $c_{\gamma_{k,n}} = c_{\beta_{k,n-1}} - 1$, then
  $$f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma) + (\gamma_{k,n}).$$
  2. If $\gamma = \beta_{k,n-1}$ and $c_{\gamma_{k,n}} < c_{\beta_{k,n-1}} - 1$, then
  $$f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - 2(\gamma) + (\gamma_{k,k}).$$
  3. If $\gamma = \gamma_{k,k}$, then
  $$f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma) + 2(\gamma_{k,n}).$$
  4. If there is no leftmost ‘(’, then
  $$f_n\boldsymbol{\alpha} = \boldsymbol{\alpha} + (\gamma_{n,n}).$$

Proposition II.31 ([10])**.** *There exists a crystal isomorphism from* $\mathrm{Kp}(\infty)$ *to* $B(\infty)$. ■

Note that the rules for e_n and f_n on $\mathrm{Kp}(\infty)$ may seem complicated, but the significance of this definition is seen in the structure of the analogous elements $\mathcal{T}(\infty)$. That is, rules (1)-(3) in these definitions are related to ways to add or remove pairs of entries $(n,\overline{n})$ in the tableau description.

We now provide some examples of the action of these operators on elements of $\mathrm{Kp}(\infty)$, and later, after the definition of the bijection between $\mathrm{Kp}(\infty)$ and $\mathcal{T}(\infty)$, we will see the connection between the action of the Kashiwara operators on these elements and the structure of the associated marginally large tableaux.

Example II.32. Let $\mathrm{Kp}(\infty)$ be of type C_3, and let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$, where

$$\boldsymbol{\alpha} = 3(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3})$$

We now consider the action of f_3 on $\boldsymbol{\alpha}$. To compute this action we first determine the 3-signature of $\boldsymbol{\alpha}$.

	$c_{\gamma_{1,1}}$	$c_{\beta_{1,2}}$	$c_{\gamma_{1,3}}$	$c_{\gamma_{1,1}}$	$c_{\gamma_{2,2}}$	$c_{\beta_{2,2}}$	$c_{\gamma_{2,3}}$	$c_{\gamma_{2,2}}$	$c_{\gamma_{3,3}}$
$S_3(\boldsymbol{\alpha}) =$	$))$	$((($	$))$	$(($	$)$		$)$	$($	$)$
$S_3^c(\boldsymbol{\alpha}) =$	$))$	$($							

The leftmost '(' in $S_3^c(\boldsymbol{\alpha})$ comes from a $(\beta_{1,2})$, and $c_{\gamma_{1,3}} = c_{\beta_{1,2}} - 1$, so this an example of case (1) for f_n from the rule in Definition II.30. Thus, the action of f_n will remove a $(\beta_{1,2})$ and add a $(\gamma_{1,3})$ to $\boldsymbol{\alpha}$:

$$f_3\boldsymbol{\alpha} = 2(\beta_{1,2}) + 3(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

Now, we consider applying e_3 to $f_3\boldsymbol{\alpha}$, which should return us to $\boldsymbol{\alpha}$. Explicitly, we compute the n-signature of $f_3\boldsymbol{\alpha}$ and look for the rightmost ‘)’.

	$c_{\gamma_{1,1}}$	$c_{\beta_{1,2}}$	$c_{\gamma_{1,3}}$	$c_{\gamma_{1,1}}$	$c_{\gamma_{2,2}}$	$c_{\beta_{2,2}}$	$c_{\gamma_{2,3}}$	$c_{\gamma_{2,2}}$	$c_{\gamma_{3,3}}$
$S_3(f_3\boldsymbol{\alpha}) =$	))	((	)))	((	)		)	(	)
$S_3^c(f_3\boldsymbol{\alpha}) =$	))		)						

Here the rightmost ‘)’ comes from $(\gamma_{1,3})$ and $c_{\gamma_{1,3}} = c_{\beta_{1,2}} + 1$, this is case (1) for e_n from Definition II.30, so the action of e_n removes a $(\gamma_{1,3})$ and adds $(\beta_{1,2})$ to $f_3\boldsymbol{\alpha}$:

$$e_3 f_3 \boldsymbol{\alpha} = 3(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}) = \boldsymbol{\alpha}.$$

Example II.33. Let $\mathrm{Kp}(\infty)$ be of type C_3, and let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$, where

$$\boldsymbol{\alpha} = 4(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

We now consider the action of f_3 on $\boldsymbol{\alpha}$. To compute this action we first determine the 3-signature of $\boldsymbol{\alpha}$.

	$c_{\gamma_{1,1}}$	$c_{\beta_{1,2}}$	$c_{\gamma_{1,3}}$	$c_{\gamma_{1,1}}$	$c_{\gamma_{2,2}}$	$c_{\beta_{2,2}}$	$c_{\gamma_{2,3}}$	$c_{\gamma_{2,2}}$	$c_{\gamma_{3,3}}$
$S_3(\boldsymbol{\alpha}) =$	))	((((	))	((	)		)	(	)
$S_3^c(\boldsymbol{\alpha}) =$	))	((							

The leftmost ‘(’ in $S_3^c(\boldsymbol{\alpha})$ comes from a $(\beta_{1,2})$, and $c_{\gamma_{1,3}} < c_{\beta_{1,2}} - 1$, so this an example of case (2) for f_n from the rule in Definition II.30. Thus, the action of f_n will remove two $(\beta_{1,2})$ and add a $(\gamma_{1,1})$ to $\boldsymbol{\alpha}$:

$$f_3 \boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 3(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

Now, we consider applying e_3 to $f_3\boldsymbol{\alpha}$, which should return us to $\boldsymbol{\alpha}$. Explicitly, we compute the n-signature of $f_3\boldsymbol{\alpha}$ and look for the rightmost ‘)’.

	$c_{\gamma_{1,1}}$	$c_{\beta_{1,2}}$	$c_{\gamma_{1,3}}$	$c_{\gamma_{1,1}}$	$c_{\gamma_{2,2}}$	$c_{\beta_{2,2}}$	$c_{\gamma_{2,3}}$	$c_{\gamma_{2,2}}$	$c_{\gamma_{3,3}}$
$S_3(f_3\boldsymbol{\alpha}) =$	)))	((	))	(((	)		)	(	)
$S_3^c(f_3\boldsymbol{\alpha}) =$	)))			(					

Here the rightmost ‘)’ comes from $(\gamma_{1,1})$ so this is case (3) for e_n from Definition II.30, so the action of e_3 removes one $(\gamma_{1,1})$ and adds two $(\beta_{1,2})$ to $f_3\boldsymbol{\alpha}$:

$$e_3 f_3\boldsymbol{\alpha} = 4(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2} + (\gamma_{2,3}) + (\gamma_{3,3}) = \boldsymbol{\alpha}.$$

Example II.34. Let $\mathrm{Kp}(\infty)$ be of type C_3, and let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$, where

$$\boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 3(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3})$$

We now consider the action of f_3 on $\boldsymbol{\alpha}$. To compute this action we first determine the 3-signature of $\boldsymbol{\alpha}$.

	$c_{\gamma_{1,1}}$	$c_{\beta_{1,2}}$	$c_{\gamma_{1,3}}$	$c_{\gamma_{1,1}}$	$c_{\gamma_{2,2}}$	$c_{\beta_{2,2}}$	$c_{\gamma_{2,3}}$	$c_{\gamma_{2,2}}$	$c_{\gamma_{3,3}}$
$S_3(\boldsymbol{\alpha}) =$	)))	((	))	(((	)		)	(	)
$S_3^c(\boldsymbol{\alpha}) =$	)))			(					

The leftmost ‘(’ in $S_3^c(\boldsymbol{\alpha})$ comes from a $(\gamma_{1,1})$ so this an example of case (3) for f_n from the rule in Definition II.30. Thus, the action of f_n will remove one $(\gamma_{1,1})$ and add two $(\gamma_{1,3})$ to $\boldsymbol{\alpha}$:

$$f_3\boldsymbol{\alpha} = 2(\beta_{1,2}) + 4(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

Now, we consider applying e_3 to $f_3\boldsymbol{\alpha}$, which should return us to $\boldsymbol{\alpha}$. Explicitly, we compute the n-signature of $f_3\boldsymbol{\alpha}$ and look for the rightmost ')'.

$$\begin{array}{lccccccccc} & c_{\gamma_{1,1}} & c_{\beta_{1,2}} & c_{\gamma_{1,3}} & c_{\gamma_{1,1}} & c_{\gamma_{2,2}} & c_{\beta_{2,2}} & c_{\gamma_{2,3}} & c_{\gamma_{2,2}} & c_{\gamma_{3,3}} \\ S_3(f_3\boldsymbol{\alpha}) = &)) & ((&)))) & ((&) & &) & (&) \\ S_3^c(f_3\boldsymbol{\alpha}) = &)) & &)) & & & & & & \end{array}$$

Here the rightmost ')' comes from $(\gamma_{1,3})$ and $c_{\gamma_{1,3}} > c_{\beta_{1,2}} + 1$, this is case (2) for e_n from Definition II.30, so the action of e_n removes two $(\gamma_{1,3})$ and adds $(\gamma_{1,1})$ to $f_3\boldsymbol{\alpha}$:

$$e_3 f_3\boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 3(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}) = \boldsymbol{\alpha}.$$

Example II.35. Let $\mathrm{Kp}(\infty)$ be of type C_3, and let $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$, where

$$\boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

We now consider the action of f_3 on $\boldsymbol{\alpha}$. To compute this action we first determine the 3-signature of $\boldsymbol{\alpha}$.

$$\begin{array}{lccccccccc} & c_{\gamma_{1,1}} & c_{\beta_{1,2}} & c_{\gamma_{1,3}} & c_{\gamma_{1,1}} & c_{\gamma_{2,2}} & c_{\beta_{2,2}} & c_{\gamma_{2,3}} & c_{\gamma_{2,2}} & c_{\gamma_{3,3}} \\ S_3(\boldsymbol{\alpha}) = &)) & ((&)) & ((&) & &) & (&) \\ S_3^c(\boldsymbol{\alpha}) = &)) & & & & & & & & \end{array}$$

In this case, there is no '(' in $S_3^c(\boldsymbol{\alpha})$ so this is an example of case (4) for f_n from the rule in Definition II.30. Thus, the action of f_n will simply add one $(\gamma_{3,3})$ to $\boldsymbol{\alpha}$:

$$f_3\boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + 2(\gamma_{3,3}).$$

Now, we consider applying e_3 to $f_3\boldsymbol{\alpha}$, which should return us to $\boldsymbol{\alpha}$. Explicitly, we compute the n-signature of $f_3\boldsymbol{\alpha}$ and look for he rightmost ')'.

$$\begin{array}{rccccccccc} & c_{\gamma_{1,1}} & c_{\beta_{1,2}} & c_{\gamma_{1,3}} & c_{\gamma_{1,1}} & c_{\gamma_{2,2}} & c_{\beta_{2,2}} & c_{\gamma_{2,3}} & c_{\gamma_{2,2}} & c_{\gamma_{3,3}} \\ S_3(f_3\boldsymbol{\alpha}) = &)) & ((&)) & ((&) & &) & (&)) \\ S_3^c(f_3\boldsymbol{\alpha}) = &)) & & & & & & & &) \end{array}$$

Here the rightmost ')' comes from $(\gamma_{3,3})$ making this a situation of case (4) for e_n from Definition II.30, so the action of e_n simply removes one $(\gamma_{3,3})$ from $f_3\boldsymbol{\alpha}$:

$$e_3 f_3\boldsymbol{\alpha} = 2(\beta_{1,2}) + 2(\gamma_{1,3}) + 2(\gamma_{1,1}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}) = \boldsymbol{\alpha}.$$

CHAPTER III

THE ISOMORPHISM

Here we present and prove the isomorphism between $\mathrm{Kp}(\infty)$ and $\mathcal{T}(\infty)$. The isomorphism, Ψ, is given as a reversible algorithm to construct an element of $\mathrm{Kp}(\infty)$ from an element of $\mathcal{T}(\infty)$. We prove this result by first showing a lemma to establish that the isomorphism holds on a subset of $\mathcal{T}(\infty)$ where all relevant entries occur in a single row. Combining the result of this lemma with the factorization of the bracketing sequences by rows we can show that Ψ is in fact a crystal isomorphism between $\mathcal{T}(\infty)$ and $B(\infty)$.

Theorem III.1. *Define* $\Psi\colon \mathcal{T}(\infty) \longrightarrow \mathrm{Kp}(\infty)$ *by the following process. Let* $T \in \mathcal{T}(\infty)$ *be a marginally large tableaux of type* B_n *or* C_n*, and let* $R_1, \dots, R_n$ *denote the rows of* T *starting at the top. Set* $\Psi(T) = \sum_{j=1}^{n} \Psi(R_j)$*, where* $\Psi(R_j)$ *is defined in the following way:*

If T *is of type* B_n*:*

1. *each pair* $\left(\boxed{n}, \boxed{\overline{n}}\right)$ *maps to* $2(\beta_{j,n})$*;*
2. *if* $j = n$*, send* $\boxed{\overline{j}}$ *to* $2(\beta_{n,n})$*;*
3. *the* $\boxed{0}$ *map to* $(\beta_{j,n})$*.*

If T *is of type* C_n*:*

4. *each pair* $\left(\boxed{n}, \boxed{\overline{n}}\right)$ *maps to* $(\gamma_{j,j})$*;*
5. *if* $j = n$*, send* $\boxed{\overline{j}}$ *to* $(\gamma_{n,n})$*.*

Then map the remaining boxes in either type:

6. *if* $j \neq n$*, send* $\boxed{\overline{j}}$ *to* $(\beta_{j,j}) + (\gamma_{j,j+1})$*;*
7. *each pair* $\left(\boxed{k}, \boxed{\overline{k}}\right)$*, where* $j < k < n$*, maps to* $(\beta_{j,k}) + (\gamma_{j,k+1})$*;*
8. *each remaining* $\boxed{k}$ *for* $k \in \{j+1, \dots, n\}$ *is sent to* $(\beta_{j,k-1})$*;*
9. *each remaining* $\boxed{\overline{k}}$ *for* $\overline{k} \in \{\overline{n}, \dots, \overline{j+1}\}$ *is sent to* $(\gamma_{j,k})$*.*

Then Ψ *is a crystal isomorphism.*

Before we proceed to prove Theorem III.1, we present examples of Ψ applied to both types of crystals by applying the map to some of the tableaux presented in previous examples.

Example III.2. Consider type B_3 and $i = 3$, recalling Example II.26, we had

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & \overline{3} & \overline{2} & \overline{1} & \overline{1} \\ \hline 2 & 2 & 2 & 2 & 3 & 0 & \overline{2} & \overline{2} \\ \cline{1-8} 3 & \overline{3} & \overline{3} \\ \cline{1-3} \end{array} .$$

The reading word and bracketing sequence are as follows.

$$\begin{aligned} \mathrm{read}(T) &= \overline{1}\ \overline{1}\ \overline{2}\ \overline{3}\ \ 0\ \ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \overline{2}\ \overline{2}\ \ 0\ \ 3\ \ 2\ 2\ 2\ 2\ \overline{3}\ \ \overline{3}\ \ 3 \\ \mathrm{br}_3(T) &= \quad))\ )(\qquad\qquad\qquad)(\ ((\qquad))\ ))\ ((\\ \mathrm{br}_3^c(T) &= \quad)) \qquad\qquad\qquad\qquad\qquad)\ ((\end{aligned}$$

Considering the application of f_3 to T we have,

$$f_3T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & \overline{3} & \overline{2} & \overline{1} & \overline{1} \\ \hline 2 & 2 & 2 & 2 & 2 & 3 & 0 & \overline{2} & \overline{2} \\ \cline{1-9} 3 & 0 & \overline{3} & \overline{3} \\ \cline{1-4} \end{array} .$$

If we apply the isomorphism given in Theorem III.1 to T and f_3T, we get

$$\begin{aligned} \Psi(T) = {} & 2\big((\beta_{1,1}) + (\gamma_{1,2})\big) + \big((\beta_{1,2}) + (\gamma_{1,3})\big) + (\gamma_{1,3}) + (\beta_{1,3}) \\ & + 2\big((\beta_{2,2}) + (\gamma_{2,3})\big) + (\beta_{2,3}) + (\beta_{2,2}) \\ & + 4(\beta_{3,3}) \\ = {} & 2(\beta_{1,1}) + (\beta_{1,2}) + (\beta_{1,3}) + 2(\gamma_{1,3}) + 2(\gamma_{1,2}) + 3(\beta_{2,2}) + (\beta_{2,3}) + 2(\gamma_{2,3}) + 4(\beta_{1,3}), \end{aligned}$$

and

$$\begin{aligned} \Psi(f_3T) = {} & 2\big((\beta_{1,1}) + (\gamma_{1,2})\big) + \big((\beta_{1,2}) + (\gamma_{1,3})\big) + (\gamma_{1,3}) + (\beta_{1,3}) \\ & + 2\big((\beta_{2,2}) + (\gamma_{2,3})\big) + (\beta_{2,3}) + (\beta_{2,2}) \\ & + 4(\beta_{3,3}) + (\beta_{3,3}) \\ = {} & 2(\beta_{1,1}) + (\beta_{1,2}) + (\beta_{1,3}) + 2(\gamma_{1,3}) + 2(\gamma_{1,2}) + 3(\beta_{2,2}) + (\beta_{2,3}) + 2(\gamma_{2,3}) + 5(\beta_{1,3}). \end{aligned}$$

The bracketing sequence on Kostant partitions is

$$\begin{array}{rcccccccc} & c_{\beta_{1,3}} & 2c_{\beta_{1,2}} & 2c_{\gamma_{1,3}} & c_{\beta_{1,3}} & c_{\beta_{2,3}} & 2c_{\beta_{2,2}} & 2c_{\gamma_{2,3}} & c_{\beta_{3,3}} \\ S_3(\Psi(T)) = &) & ((&)))) & (& (& ((((((&)))) &)))) \\ S_3^c(\Psi(T)) = &) & &)) & & & & & \end{array}$$

Since after cancellation the bracket sequence $S_3(\Psi(T))$ contains no left brackets, by Definition II.30, $f_3\Psi(T) = \Psi(T) + (\beta_{3,3})$, which agrees with the computed result for $\Psi(f_3(T))$.

Example III.3. Let $T \in \mathcal{T}(\infty)$ be of type C_3 where

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & \overline{3} & \overline{3} & \overline{2} & \overline{2} \\ \hline 2 & 2 & 2 & 3 & \overline{3} & \overline{3} \\ \cline{1-6} 3 & \overline{3} \\ \cline{1-2} \end{array} \; .$$

The reading word and bracketing sequence are as follows.

$$\begin{array}{rccccccccccccccccccccccccc} \mathrm{read}_{\mathrm{ME}}(T) = & \overline{2} & \overline{2} & \overline{3} & \overline{3} & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \overline{3} & \overline{3} & 3 & 2 & 2 & 2 & \overline{3} & 3 \\ \mathrm{br}_3(T) = & & &) &) & (& (& (& (& & & & & & & & & &) &) & (& & & &) & (\\ \mathrm{br}_3^c(T) = & & &) &) & (& (& & & & & & & & & & & & & & & & & & & (\end{array}$$

Considering the application of f_3 to T we have,

$$f_3T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & \overline{3} & \overline{3} & \overline{3} & \overline{2} & \overline{2} \\ \hline 2 & 2 & 2 & 3 & \overline{3} & \overline{3} \\ \cline{1-6} 3 & \overline{3} \\ \cline{1-2} \end{array} \; .$$

Now by applying the isomorphism from Theorem III.1 to T and f_3T we get

$$\begin{aligned} \Psi(T) &= 2(\gamma_{1,1}) + 2\big((\beta_{1,2}) + (\gamma_{1,3})\big) + 2(\beta_{1,2}) \\ &\quad + (\gamma_{2,2}) + (\gamma_{2,3}) \\ &\quad + (\gamma_{3,3}) \\ &= 4(\beta_{1,2}) + 2(\gamma_{1,1}) + 2(\gamma_{1,3}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}), \text{ and} \\ \Psi(f_3T) &= 3(\gamma_{1,1}) + 2\big((\beta_{1,2}) + (\gamma_{1,3})\big) \\ &\quad + (\gamma_{2,2}) + (\gamma_{2,3}) \end{aligned}$$

$$+(\gamma_{3,3})$$

$$= 2(\beta_{1,2}) + 3(\gamma_{1,1}) + 2(\gamma_{1,3}) + (\gamma_{2,2}) + (\gamma_{2,3}) + (\gamma_{3,3}).$$

Note that these are the same Kostant partitions we worked with in Example II.33. By what we saw previously applying f_3 to $\Psi(T)$ will remove two $(\beta_{1,2})$ and add a $(\gamma_{1,1})$, which agrees with $\Psi(f_3T)$. That is $f_3\Psi(T) = \Psi(f_3T)$.

The proof of Theorem III.1 will occupy the rest of this section. Denote by $e_i^{\mathcal{T}}$ and $f_i^{\mathcal{T}}$ the Kashiwara operators on $\mathcal{T}(\infty)$ from Definition II.22, and by e_i^{Kp} and f_i^{Kp} those on $\mathrm{Kp}(\infty)$ from Definition II.30. In order to facilitate the proof of Theorem III.1 we first prove for tableaux where all relevant entries are in a single row.

Lemma III.4. *Consider $\mathcal{T}(\infty)$ of type B_n or C_n. Fix $i \in I$ and a row index j. Let $T \in \mathcal{T}(\infty)$ be such that the only unshaded boxes appearing in T occur in row j. If the leftmost '(' in $\mathrm{br}_i^c(T)$ comes from R_j, then $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$. Furthermore, if $j < i$, then $\mathrm{br}_i^c(R_j)$ and $S_i^c\big(\Psi(R_j)\big)$ have the same number of left and right brackets.*

Proof. For row $j < i$ we show that $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$ by establishing the stronger result that the i-signatures are equivalent, that is $\mathrm{br}_i^c(T) = S_i^c(\Psi(T))$ which implies $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$. Note that when $j > i+1$ both i-signatures are empty. When $j = i$ or $j = i+1$, the i-signatures are not equivalent between the two realizations. However, we are able to readily establish that $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$ by considering that the behavior of f_i simplifies when $j = i$ or $j = i+1$. We handle these cases separately.

First consider $i \in \{1,\dots,n-1\}$, and row R_j for $j < i$. We are only interested in the boxes $\boxed{i}$, $\boxed{i+1}$, $\boxed{\overline{\imath+1}}$, and $\boxed{\bar{\imath}}$. This follows from considering which boxes result in brackets in $\mathrm{br}_i(T)$ or in $S_i\big(\Psi(T)\big)$. Now, assume row j of T has p, q, r, and s boxes of $\overline{\imath+1}$, $i+1$, i, and $\bar{\imath}$ respectively:

$$R_j = \underbrace{\boxed{i}\,\boxed{\cdots}\,\boxed{i}}_{r}\,\underbrace{\boxed{i+1}\,\boxed{\cdots}\,\boxed{i+1}}_{q}\,\boxed{\cdots}\,\underbrace{\boxed{\overline{\imath+1}}\,\boxed{\cdots}\,\boxed{\overline{\imath+1}}}_{p}\,\underbrace{\boxed{\bar{\imath}}\,\boxed{\cdots}\,\boxed{\bar{\imath}}}_{s}$$

Define $\Psi(R_j) = \sum_{(\beta)\in\mathcal{R}} c_\beta(\beta)$. The general bracketing sequences for both are given here.

$$\mathrm{br}_i(R_j) =)^s\ (^p\ )^q\ (^r\ , \qquad S_i\big(\Psi(R_j)\big) = \underbrace{)\cdots)}_{c_{\beta_{j,i}}}\ \underbrace{(\cdots(}_{c_{\beta_{j,i-1}}}\ \underbrace{)\cdots)}_{c_{\gamma_{j,i}}}\ \underbrace{(\cdots(}_{c_{\gamma_{j,i+1}}}.$$

Case 1: $p > q$, $r > s$ and $1 \le j < i < n$. By the definition of Ψ,

$$\begin{aligned}\Psi(R_j) &= q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q)(\gamma_{j,i+1}) + (r-s)(\beta_{j,i-1}) + s(\beta_{j,i} + \gamma_{j,i+1}) \\ &= (r-s)(\beta_{j,i-1}) + s(\beta_{j,i}) + q(\beta_{j,i+1}) + (s+p-q)(\gamma_{j,i+1}) + q(\gamma_{j,i+2}).\end{aligned}$$

Calculating the action of $f_i^{\mathcal{T}}$ on R_j gives

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{i+1}}\boxed{\cdots}\boxed{\overline{i+1}}}_{p-1}\underbrace{\boxed{\bar{\imath}}\boxed{\cdots}\boxed{\bar{\imath}}}_{s+1}.$$

Then

$$\begin{aligned}\Psi(f_i^{\mathcal{T}} R_j) &= q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q-1)(\gamma_{j,i+1}) + (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i} + \gamma_{j,i+1}) \\ &= (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i}) + q(\beta_{j,i+1}) + (s+p-q)(\gamma_{j,i+1}) + q(\gamma_{j,i+2}).\end{aligned}$$

We now apply the operator f_i^{Kp} to $\Psi(R_j)$ to show equivalence. In $S_i^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\beta_{j,i-1}$, so

$$\begin{aligned}f_i^{\mathrm{Kp}}\Psi(R_j) &= (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (s+p-q)(\gamma_{j,i+1}) \\ &= \Psi(f_i^{\mathcal{T}} R_j).\end{aligned}$$

Furthermore,

$$\mathrm{br}_i(R_j) =)^s\ (^{p-q}\ (^r \qquad \text{and} \qquad S_i\big(\Psi(R_j)\big) =)^s\ (^{r-s}\ (^{s+p-q},$$

so both $\mathrm{br}_i^c(R_j)$ and $S_i^c\big(\Psi(R_j)\big)$ have s ')' and $r+p-q$ '('.

Case 2: $p > q$, $r \le s$, and $1 \le j < i < n$. By the definition of Ψ,

$$\Psi(R_j) = q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q)(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}) + r(\beta_{j,i} + \gamma_{j,i+1})$$

$$= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r+p-q)(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}).$$

By the definition of $f_i^{\mathcal{T}}$, we have

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{i+1}}\boxed{\cdots}\boxed{\overline{i+1}}}_{p-1}\underbrace{\boxed{\bar{i}}\boxed{\cdots}\boxed{\bar{i}}}_{s+1}.$$

Then

$$\begin{aligned}\Psi(f_i^{\mathcal{T}} R_j) &= q(\beta_{j,i+1} + \gamma_{j,i+2}) + (p-q-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}) + r(\beta_{j,i} + \gamma_{j,i+1}) \\ &= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r+p-q-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}).\end{aligned}$$

On the other hand, in $S_i^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\gamma_{j,i+1}$, so

$$\begin{aligned}f_i^{\mathrm{Kp}}\Psi(R_j) &= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r+p-q-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}) \\ &= \Psi(f_i^{\mathcal{T}} R_j).\end{aligned}$$

Furthermore,

$$\mathrm{br}_i(R_j) =)^s \; (^{p-q} \; (^r \qquad \text{and} \qquad S_i\big(\Psi(R_j)\big) =)^r \;)^{s-r} \; (^{r+p-q},$$

so both $\mathrm{br}_i^c(R_j)$ and $S_i^c\big(\Psi(R_j)\big)$ have s ')' and $r+p-q$ '('.

Case 3: $p \le q$, $r > s$, and $1 \le j < i < n$. By the definition of Ψ,

$$\begin{aligned}\Psi(R_j) &= p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p)(\beta_{j,i}) + (r-s)(\beta_{j,i-1}) + s(\beta_{j,i} + \gamma_{j,i+1}) \\ &= (r-s)(\beta_{j,i-1}) + (s+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}).\end{aligned}$$

By the definition of $f_i^{\mathcal{T}}$, we have

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r-1}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q+1}\boxed{\cdots}\underbrace{\boxed{\overline{i+1}}\boxed{\cdots}\boxed{\overline{i+1}}}_{p}\underbrace{\boxed{\bar{i}}\boxed{\cdots}\boxed{\bar{i}}}_{s}.$$

Then

$$\Psi(f_i^{\mathcal{T}} R_j) = p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p+1)(\beta_{j,i}) + (r-s-1)(\beta_{j,i-1}) + s(\beta_{j,i} + \gamma_{j,i+1})$$
$$= (r-s-1)(\beta_{j,i-1}) + (s+q-p+1)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}).$$

On the other hand, in $S_i^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\beta_{j,i-1}$, so

$$f_i^{\mathrm{Kp}}\Psi(R_j) = (r-s-1)(\beta_{j,i-1}) + (s+q-p+1)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1})$$
$$= \Psi(f_i^{\mathcal{T}} R_j).$$

Furthermore,

$$\mathrm{br}_i(R_j) =)^s\ )^{q-p}\ (^r \quad \text{and} \quad S_i\big(\Psi(R_j)\big) =)^{s+q-p}\ (^{r-s}\ (^s,$$

so both $\mathrm{br}_i^c(R_j)$ and $S_i^c\big(\Psi(T)\big)$ have $s+q-p$ ')' and r '('.

Case 4: $p \leq q$, $r \leq s$, and $1 \leq j < i < n$. By the definition of Ψ,

$$\Psi(R_j) = p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p)(\beta_{j,i}) + (s-r)(\gamma_{j,i}) + r(\beta_{j,i} + \gamma_{j,i+1})$$
$$= (r+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + r(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}).$$

If $r = 0$, then f_i will act on the rightmost $\boxed{i}$ in R_i of T (see Case 6 for details on this situation). When $r > 0$ by the definition of $f_i^{\mathcal{T}}$, we have

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\,\boxed{\cdots}\,\boxed{i}}_{r-1}\,\underbrace{\boxed{i+1}\,\boxed{\cdots}\,\boxed{i+1}}_{q+1}\,\boxed{\cdots}\,\underbrace{\boxed{\overline{i+1}}\,\boxed{\cdots}\,\boxed{\overline{i+1}}}_{p}\,\underbrace{\boxed{\bar{\imath}}\,\boxed{\cdots}\,\boxed{\bar{\imath}}}_{s}\,.$$

Then

$$\Psi(f_i^{\mathcal{T}}(R_j) = p(\beta_{j,i+1} + \gamma_{j,i+2}) + (q-p+1)(\beta_{j,i}) + (s-r+1)(\gamma j, i) + (r-1)(\beta_{j,i} + \gamma_{j,i+1})$$
$$= (r+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + (r-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}).$$

On the other hand, in $S_i^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\gamma_{j,i+1}$, so

$$f_i^{\mathrm{Kp}}\Psi(R_j) = (r+q-p)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + (r-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i})$$
$$= \Psi(f_i^{\mathcal{T}} R_j).$$

Furthermore

$$\mathrm{br}_i(R_j) =)^s\ )^{q-p}\ (^r \quad \text{and} \quad S_i\big(\Psi(R_j)\big) =)^{r+q-p}\ )^{s-r}\ (^r,$$

so both $\mathrm{br}_i^c(R_j)$ and $S_i^c\big(\Psi(T)\big)$ have $s+q-p$ ')' and r '('.

We now establish the result for $i \in \{1,\dots,n-1\}$ and row $j = i$. The general bracketing sequences for both are given here:

$$\mathrm{br}_i(R_i) =)^s\ (^p\ )^q\ (^r\ , \qquad\qquad S_i\big(\Psi(R_i)\big) = \underbrace{)\cdots)}_{c_{\beta_{i,i}}}\ .$$

Note that based on Ψ the only positive root which could come from $\Psi(R_i)$ and contribute to the i-signature is $(\beta_{i,i})$. Since there is no '(' in $S_i\big(\Psi(R_i)\big)$, the action of f_i^{Kp} will always be to add $(\beta_{i,i})$ following Definition II.30. It now remains to show the action of $f_i^{\mathcal{T}}$ on R_i does the same change.

Case 5: $p > q$, and $1 \le j = i < n$. If $p > q$, then the leftmost '(' comes from an $\boxed{\overline{i+1}}$, so $f_i^{\mathcal{T}} R_i$ sends an $\boxed{\overline{i+1}}$ to a $\boxed{\bar{i}}$. Since $p > q$, removing a $\boxed{\overline{i+1}}$ does not change the number of $\Big(\boxed{i+1}, \boxed{\overline{i+1}}\Big)$ pairs. According to Ψ we then have that $\Psi(f_i^{\mathcal{T}} R_i) = \Psi(R_i) + (\beta_{i,i}) = f_i^{\mathrm{Kp}}\Psi(R_i)$.

Case 6: $p \le q$, and $1 \le j = i < n$. If $p \le q$, then the leftmost '(' comes from a $\boxed{i}$, so $f_i^{\mathcal{T}} R_i$ sends an $\boxed{i}$ to an $\boxed{i+1}$. Since $p \le q$, addition of an $\boxed{i+1}$ does not change the number of $\Big(\boxed{i+1}, \boxed{\overline{i+1}}\Big)$ pairs. So, again by Ψ we have that $\Psi(f_i^{\mathcal{T}} R_i) = \Psi(R_i) + (\beta_{i,i}) = f_i^{\mathrm{Kp}}\Psi(R_i)$.

Case 7: $2 \le j = i+1 \le n$. By the marginally large condition, the leftmost '(' in the $\mathrm{br}_i^c(T)$ cannot come from an box in the $(i+1)$-st row. For T which are in this case, the action of $f_i^{\mathcal{T}}$ will always be to promote a $\boxed{i}$ in the i-th row to a $\boxed{i+1}$. Following Ψ, this says that

$f_i^{\mathrm{Kp}}\Psi(R_{i+1}) = \Psi(R_{i+1}) + (\beta_{i,i})$. Then the sequence $S_i\big(\Psi(R_{i+1})\big)$ will be empty, as none of the relevant positive roots map from entries in the $(i+1)$-st row. Combining these two remarks then gives that

$$f_i^{\mathcal{T}}(R_j) = \Psi(R_j) + (\beta_{i,i}) = f_i^{\mathrm{Kp}}\Psi(R_j)$$

as desired.

For the $i = n$ situation we deal with T of type B_n and those of type C_n separately. First we consider T to be of type B_n. Here we are only interested in the $\boxed{n}$, $\boxed{0}$, and $\boxed{\bar{n}}$ boxes. We now assume row j of T has p $\boxed{\bar{n}}$ boxes, z $\boxed{0}$ boxes, and q $\boxed{n}$ boxes.

$$R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q}\underbrace{\boxed{0}}_{z}\underbrace{\boxed{\bar{n}}\boxed{\cdots}\boxed{\bar{n}}}_{p}\,.$$

The general bracketing sequences for both are given here:

$$\mathrm{br}_i(R_j) =)^{2p}\ )^{z}\ (^{z}\ (^{2q}\ , \qquad\qquad S_i\big(\Psi(T)\big) = \underbrace{)\cdots)}_{c_{\beta_{j,n}}}\ \underbrace{(\cdots(}_{2c_{\beta_{j,n-1}}}\ \underbrace{)\cdots)}_{2c_{\gamma_{j,n}}}\ \underbrace{(\cdots(}_{c_{\beta_{j,n}}}\,.$$

When row $j < n$, we have four cases.

Case 8: type B_n, $p \geq q$, $z = 0$, and $1 \leq j < i = n$. By the definition of Ψ,

$$\Psi(R_j) = 2q(\beta_{j,n}) + (p-q)(\gamma_{j,n}).$$

If $q = 0$, then f_n will act on the $\boxed{n}$ in R_n of T (see Case 12 for more details in this situation). When $q > 0$ by the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}}R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{0}}_{1}\underbrace{\boxed{\bar{n}}\boxed{\cdots}\boxed{\bar{n}}}_{p}\,.$$

In this situation, with $p \geq q$, when applying f_n a $\big(\boxed{n},\boxed{\bar{n}}\big)$ pair is removed, giving an additional $\boxed{0}$ and an unpaired $\boxed{\bar{n}}$. Then

$$\Psi(f_n^{\mathcal{T}}R_j) = 2(q-1)(\beta_{j,n}) + (\beta_{j,n}) + (p-q+1)(\gamma_{j,n}),$$

$$= (2q-1)(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\beta_{j,n}$, so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (2q-1)(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}) = \Psi(f_n^{\mathfrak{T}} R_j).$$

Furthermore,

$$\mathrm{br}_n(R_j) =)^{2p}\ (^{2q} \qquad \text{and} \qquad S_n\big(\Psi(R_j)\big) =)^{2q}\ )^{2(p-q)}\ (^{2q},$$

so both $\mathrm{br}_n^c(R_j)$ and $S_n^c\big(\Psi(R_j)\big)$ have $2p$ ')' and $2q$ '('.

Case 9: type B_n, $p < q$, $z = 0$, and $1 \le j < i = n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + 2p(\beta_{j,n}).$$

By the definition of $f_n^{\mathfrak{T}}$, we have

$$f_n^{\mathfrak{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{0}}_{1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p}\,.$$

Note that because $p < q$ and $z = 0$ this operation neither adds nor removes a pair of $\big(\boxed{n}, \boxed{\overline{n}}\big)$. This operation reduces the number of unpaired $\boxed{n}$ by one, and adds a $\boxed{0}$. Then

$$\begin{aligned}\Psi(f_n^{\mathfrak{T}} R_j) &= (q-p-1)(\beta_{j,n-1}) + (\beta_{j,n}) + 2(p)(\beta_{j,n}) \\ &= (q-p-1)(\beta_{j,n-1}) + (2p+1)(\beta_{j,n}).\end{aligned}$$

On the other hand, in $S_i^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\beta_{j,n-1}$, so,

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+1)(\beta_{j,j}) = \Psi(f_n^{\mathfrak{T}} R_j).$$

Furthermore,

$$\mathrm{br}_n(R_j) =)^{2p}\ (^{2q} \qquad \text{and} \qquad S_n\big(\Psi(R_j)\big) =)^{2p}\ (^{2(q-p)}\ (^{2p},$$

so both $\mathrm{br}_n^c(R_j)$ and $S_n^c\big(\Psi(R_j)\big)$ have $2p$ ')' and $2q$ '('.

Case 10: type B_n, $p \geq q$, $z = 1$, and $1 \leq j < i = n$. By the definition of Ψ,

$$\Psi(R_j) = (2q+1)(\beta_{j,n}) + (p-q)(\gamma_{j,n}).$$

By the definition of $f_n^{\mathfrak{T}}$, we have

$$f_n^{\mathfrak{T}} R_j = \underbrace{\begin{array}{|c|c|c|}\hline n & \cdots & n \\ \hline\end{array}}_{q}\underbrace{\begin{array}{|c|}\hline 0 \\ \hline\end{array}}_{0}\underbrace{\begin{array}{|c|c|c|}\hline \overline{n} & \cdots & \overline{n} \\ \hline\end{array}}_{p+1}.$$

Here, with $p \geq q$ and $z = 1$, this operation removes the $\boxed{0}$ and adds an unpaired $\boxed{\overline{n}}$. Then

$$\Psi(f_n^{\mathfrak{T}} R_j) = 2q(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\beta_{j,n}$, so,

$$f_n^{\mathrm{Kp}}\Psi(R_j) = 2q(\beta_{j,n}) + (p-q+1)(\gamma_{j,n}) = \Psi(f_n^{\mathfrak{T}} R_j).$$

Furthermore,

$$\mathrm{br}_n(R_j) =)^{2p}\ )\ (\ (^{2q} \qquad \text{and} \qquad S_n\big(\Psi(R_j)\big) =)^{2q+1}\ )^{2(p-q)}\ (^{2q+1},$$

so both $\mathrm{br}_n^c(R_j)$ and $S_n^c\big(\Psi(R_j)\big)$ have $2p+1$ ')' and $2q+1$ '('.

Case 11: type B_n, $p < q$, $z = 1$ and $1 \leq j < i = n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + (2p+1)(\beta_{j,n}).$$

By the definition of $f_i^{\mathfrak{T}}$, we have

$$f_n^{\mathfrak{T}} R_j = \underbrace{\begin{array}{|c|c|c|}\hline n & \cdots & n \\ \hline\end{array}}_{q}\underbrace{\begin{array}{|c|}\hline 0 \\ \hline\end{array}}_{0}\underbrace{\begin{array}{|c|c|c|}\hline \overline{n} & \cdots & \overline{n} \\ \hline\end{array}}_{p+1}.$$

Here, with $p < q$ and $z = 1$, this operation adds a $(\boxed{n}, \boxed{\bar{n}})$ by pairing an unpaired $\boxed{n}$ and removes the $\boxed{0}$. Then

$$\Psi(f_n^{\mathfrak{T}} R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+2)(\beta_{j,n})$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$, the leftmost '(' corresponds to $\beta_{j,n-1}$, so,

$$f_n^{\mathrm{Kp}} \Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (2p+2)(\beta_{j,n}) = \Psi(f_n^{\mathfrak{T}} R_j).$$

Furthermore,

$$\mathrm{br}_n(R_j) =)^{2p}\ )\ (\ (^{2q} \qquad \text{and} \qquad S_n\big(\Psi(R_j)\big) =)^{2p+1}\ (^{2(q-p)}\ (^{2p+1},$$

so both $\mathrm{br}_n^c(R_j)$ and $S_n^c\big(\Psi(R_j)\big)$ have $2p+1$ ')' and $2q+1$ '('.

Now we consider the n-signature for type B_n for row $j = n$. The n-signatures are obtained from the number of relevant boxes in R_n, and the weights on the positive roots in $\Psi(R_n)$. The general bracketing sequences for both are given here.

$$\mathrm{br}_n(R_n) =)^{2p}\ )^z\ (^z\ (^{2q}, \qquad\qquad S_n\big(\Psi(R_n)\big) = \underbrace{)\cdots)}_{c_{\beta_{n,n}}}.$$

Note that based on Ψ the only positive root which could come from R_n and contribute to the n-signature is $(\beta_{n,n})$. Since there is no '(' in $S_n\big(\Psi(R_n)\big)$, the action of f_n^{Kp} will always be to add $(\beta_{n,n})$ following Definition II.30. It now remains to show the action of $f_i^{\mathfrak{T}}$ on R_i does the same.

Case 12: type B_n and $j = i = n$. If $z = 1$, then the leftmost '(' comes from an $\boxed{0}$, so $f_n^{\mathfrak{T}}(R_i)$ sends an $\boxed{0}$ to a $\boxed{\bar{n}}$. According to Ψ we then have that

$$\Psi(f_n^{\mathfrak{T}} R_n) = \Psi(R_n) + (\beta_{n,n}) = f_n^{\mathrm{Kp}} \Psi(R_n).$$

If $z = 0$, then the leftmost '(' comes from an $\boxed{n}$, so $f_n^{\mathcal{T}} R_n$ sends an $\boxed{n}$ to an $\boxed{0}$. Again by Ψ we have that

$$\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\beta_{n,n}) = f_n^{\mathrm{Kp}} \Psi(R_n).$$

Now, consider T to be of type C_n. Here we are only interested in the $\boxed{n}$ and $\boxed{\overline{n}}$ boxes. We now assume row j of T has p $\boxed{\overline{n}}$ boxes and q $\boxed{n}$ boxes.

$$R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p}.$$

The n-signatures are obtained from the number of relevant boxes in R_j, and the weights on the positive roots in $\Psi(R_j)$. The general bracketing sequences for both are given here.

$$\mathrm{br}_i(R_j) =)^p \ (^q, \qquad S_i\big(\Psi(R_j)\big) = \underbrace{)\cdots)}_{c_{\gamma_{j,j}}} \ \underbrace{(\cdots(}_{c_{\beta_{j,n-1}}} \ \underbrace{)\cdots)}_{c_{\gamma_{j,n}}} \ \underbrace{(\cdots(}_{c_{\gamma_{j,j}}}.$$

When row $1 \leq j < i = n$, we have three more cases.

Case 13: type C_n, $p \geq q$, and $1 \leq j < i = n$. By the definition of Ψ,

$$\Psi(R_j) = q(\gamma_{j,j}) + (p-q)(\gamma_{j,n}).$$

If $q = 0$ then f_n will act on the $\boxed{n}$ in R_n of T (see Case 16 for more details in this situation). When $q > 0$ by the definition of $f_n^{\mathcal{T}}$ we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

Note that this removes a $\left(\boxed{n}, \boxed{\overline{n}}\right)$ pair. Removing a pair in this way results in two new unpaired $\boxed{\overline{n}}$. Then,

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-1)(\gamma_{j,j}) + (p-q+2)(\gamma_{j,n}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\gamma_{j,j}$, so

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-1)(\gamma_{j,j}) + (p-q+2)(\gamma_{j,n}) = \Psi(f_n^{\mathfrak{T}} R_j).$$

Furthermore,

$$\mathrm{br}_n(R_j) =)^p\ (^q \qquad \text{and} \qquad S_n\big(\Psi(R_j)\big) =)^q\ )^{p-q}\ (^q,$$

so both $\mathrm{br}_n^c(R_j)$ and $S_n^c\big(\Psi(R_j)\big)$ have p ')' and q '('.

Case 14: type C_n, $q > p+1$, and $1 \le j < i = n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + p(\gamma_{j,j}).$$

By the definition of $f_n^{\mathfrak{T}}$, we have

$$f_n^{\mathfrak{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\bar{n}}\boxed{\cdots}\boxed{\bar{n}}}_{p+1}.$$

Note that because $q > p+1$ this operation adds a $\big(\boxed{n}, \boxed{\bar{n}}\big)$ pair. Addition of a pair in this way removes two unpaired $\boxed{n}$. Then

$$\Psi(f_n^{\mathfrak{T}} R_j) = (q-p-2)(\beta_{j,n-1}) + (p+1)(\gamma_{j,j}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n-1}$, and $q-p > 1$ so,

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-2)(\beta_{j,n-1}) + (p+1)(\gamma_{j,j}) = \Psi(f_n^{\mathfrak{T}} R_j).$$

Furthermore,

$$\mathrm{br}_n(R_j) =)^p\ (^q \qquad \text{and} \qquad S_n\big(\Psi(R_j)\big) =)^p\ (^{q-p}\ (^p,$$

so both $\mathrm{br}_n^c(R_j)$ and $S_n^c\big(\Psi(T)\big)$ have p ')' and q '(.'

Case 15: type C_n, $q = p+1$, and $j < i = n$. By the definition of Ψ,

$$\Psi(R_j) = (q-p)(\beta_{j,n-1}) + p(\gamma_{j,j}).$$

By the definition of $f_n^{\mathcal{T}}$, we have

$$f_n^{\mathcal{T}} R_j = \underbrace{\boxed{n}\boxed{\cdots}\boxed{n}}_{q-1}\underbrace{\boxed{\overline{n}}\boxed{\cdots}\boxed{\overline{n}}}_{p+1}.$$

Note that because $q < p$ and $q - p$ this operation neither removes or adds a $(\boxed{n}, \boxed{\overline{n}})$ pair. This operation moves a $\boxed{n}$ to a $\boxed{\overline{n}}$. Then

$$\Psi(f_n^{\mathcal{T}} R_j) = (q-p-1)(\beta_{j,n-1}) + (\gamma_{j,n}) + p(\gamma_{j,j}).$$

On the other hand, in $S_n^c\big(\Psi(R_j)\big)$ the leftmost '(' corresponds to $\beta_{j,n-1}$, and $q - p = 1$ so,

$$f_n^{\mathrm{Kp}}\Psi(R_j) = (q-p-1)(\beta_{j,n-1}) + (\gamma_{j,n}) + p(\gamma_{j,j}) = \Psi(f_n^{\mathcal{T}} R_j).$$

Furthermore,

$$\mathrm{br}_n(R_j) =)^p\ (^q \qquad \text{and} \qquad S_n\big(\Psi(R_j)\big) =)^p\ (^{q-p}\ (^p,$$

so both $\mathrm{br}_n^c(R_j)$ and $S_n^c\big(\Psi(T)\big)$ have p ')' and q '('.

We now establish the result for $j = i = n$ for crystals of type C_n. The general bracketing sequences for both are given here:

$$\mathrm{br}_n(R_n) =)^{2p}\ (^{2q}, \qquad\qquad S_n\big(\Psi(R_n)\big) = \underbrace{)\cdots)}_{c_{\gamma_{n,n}}}.$$

Case 16: type C_n, and $j = i = n$. Note that based on Ψ the only positive root which could come from $\Psi(R_n)$ and contribute to the n-signature is $(\gamma_{n,n})$. Since there is no '(' in $S_n\big(\Psi(R_n)\big)$, the action of f_n^{Kp} will always be to add $(\gamma_{n,n})$ following Definition II.30. It now remains to show the action of $f_n^{\mathcal{T}}$ on R_n does the same. The leftmost '(' comes from an $\boxed{n}$, so $f_n^{\mathcal{T}}(R_n)$ sends an $\boxed{n}$ box to a $\boxed{\overline{n}}$ box. According to Ψ we then have that $\Psi(f_n^{\mathcal{T}} R_n) = \Psi(R_n) + (\gamma_{n,n}) = f_n^{\mathrm{Kp}}\Psi(R_n)$. ■

Proof of Theorem III.1. It suffices to show that, for all i, $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$. By the definition of the bracketing sequences and of Ψ we have

$$\mathrm{br}_i(T) \text{ factors as } \mathrm{br}_i(R_1)\mathrm{br}_i(R_2)\cdots\mathrm{br}_i(R_n), \text{ and}$$

$$S_i(\Psi(T)) \text{ factors as } S_i\big(\Psi(R_1)\big)S_i\big(\Psi(R_2)\big)\cdots S_i\big(\Psi(R_n)\big).$$

Let the leftmost '(' in $\mathrm{br}_i^c(T)$ come from row R_j. If $j \leq i$ then f_i acts on R_j and modifies the corresponding factor. Since the rest of the bracketing sequence remains unchanged we have $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$ by Lemma III.4. If $j > i$ than $\mathrm{br}_i^c(T)$ has no left bracket coming from R_j by the condition of marginal largeness and f_i acts on R_i. Thus for all $T \in \mathcal{T}(\infty)$ we have $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$. ■

CHAPTER IV

STACK NOTATION

This work is a type B_n and type C_n analogue of the results for types A_n and D_n result found in [3] and [11], respectively. The type A_n result may be described within the framework of multisegments [6, 8, 12], which have the advantage of a convenient diagrammatic notation that makes the crystal structure apparent. In [11], this was extended to show that one may introduce a *stack* notation for Kostant partitions in type D_n in which the crystal structure may be read off easily. The following section describes a similar revealing *stack* notation for crystals of types B_n and C_n.

For crystals of type B_n, make the associations,

$$\beta_{j,k} = \begin{array}{c} k \\ \vdots \\ j \end{array}, \qquad \gamma_{\ell,m} = \begin{array}{c} m \\ \vdots \\ n-1 \\ n\ n \\ n-1 \\ \vdots \\ \ell \end{array},$$

where $1 \leq j \leq k \leq n$ and $1 \leq \ell < m \leq n$.

For crystals of types C_n, make the associations,

$$\beta_{j,k} = \begin{array}{c} k \\ \vdots \\ j \end{array}, \qquad \gamma_{\ell,m} = \begin{array}{c} m \\ \vdots \\ n-1 \\ n \\ n-1 \\ \vdots \\ \ell \end{array}, \qquad \gamma_{h,h} = \begin{array}{c} n \\ n-1\ n-1 \\ \vdots \\ h\ h \end{array},$$

where $1 \leq j \leq k < n$, $1 \leq \ell < m \leq n$, and $1 \leq h \leq n$.

We can now restate the sequences given in Definition II.29 in terms of this stack notation for each type.

Given $i \in I$, the elements of Φ_i^B and Φ_i^C from Definition II.29 are the positive roots for which i may be either added or removed from the top of the corresponding stack to obtain a stack for another root. For Φ_n^C, the sequence consists of the roots which n can be added or removed from the top of the stack to either obtain another positive root or those roots whose stack splits into two stacks for another root. For example removing a n from the top of the stack for $\gamma_{4,4}$ in type C_4, splits $\gamma_{4,4}$ into two $\beta_{1,3}$ stacks. That is,

$$e_4^{\mathrm{Kp}}(\gamma_{4,4}) = 2(\beta_{1,3}) \qquad \Longrightarrow \qquad e_4\left(\begin{smallmatrix} & 4 & \\ 3 & & 3 \\ 2 & & 2 \\ 1 & & 1 \end{smallmatrix}\right) = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \; \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$$

The definition of the i-signatures for $\mathrm{Kp}(\infty)$ given in II.29 involves a complicated construction. However, the complications are simplified some by considering the structure of the stack notation. Recall that we built the i-signatures on $\mathrm{Kp}(\infty)$ in terms of some particular sequences on the positive roots. We now represent these sequences using the stack notation.

For $1 \leq i < n$,

$$\Phi_i^B = \left(\begin{smallmatrix} i \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} i-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} i \\ \vdots \\ n-1 \\ n\,n \\ n-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} i+1 \\ \vdots \\ n-1 \\ n\,n \\ n-1 \\ \vdots \\ 1 \end{smallmatrix}, \cdots, \begin{smallmatrix} i \\ i-1 \end{smallmatrix}, i-1, \begin{smallmatrix} i \\ \vdots \\ n-1 \\ n\,n \\ n-1 \\ \vdots \\ i-1 \end{smallmatrix}, \begin{smallmatrix} i+1 \\ \vdots \\ n-1 \\ n\,n \\ n-1 \\ \vdots \\ i-1 \end{smallmatrix}, i \right)$$

$$\Phi_n^B = \left(\begin{smallmatrix} n \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} n-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} n\,n \\ n-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} n \\ \vdots \\ 1 \end{smallmatrix}, \cdots, \begin{smallmatrix} n \\ n-1 \end{smallmatrix}, n-1, \begin{smallmatrix} n\,n \\ n-1 \end{smallmatrix}, \begin{smallmatrix} n \\ n-1 \end{smallmatrix}, n \right)$$

$$\Phi_i^C = \left(\begin{smallmatrix} i \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} i-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} i \\ \vdots \\ n-1 \\ n \\ n-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} i+1 \\ \vdots \\ n-1 \\ n \\ n-1 \\ \vdots \\ 1 \end{smallmatrix}, \cdots, \begin{smallmatrix} i \\ i-1 \end{smallmatrix}, i-1, \begin{smallmatrix} i \\ \vdots \\ n-1 \\ n \\ n-1 \\ \vdots \\ i-1 \end{smallmatrix}, \begin{smallmatrix} i+1 \\ \vdots \\ n-1 \\ n \\ n-1 \\ \vdots \\ i-1 \end{smallmatrix}, i \right)$$

$$\Phi_n^C = \left(\begin{smallmatrix} & n & \\ n-1 & & n-1 \\ & \vdots & \\ 1 & & 1 \end{smallmatrix}, \begin{smallmatrix} n-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} n \\ n-1 \\ \vdots \\ 1 \end{smallmatrix}, \begin{smallmatrix} & n & \\ n-1 & & n-1 \\ & \vdots & \\ 1 & & 1 \end{smallmatrix}, \cdots, \begin{smallmatrix} & n & \\ n-1 & & n-1 \end{smallmatrix}, n-1, \begin{smallmatrix} n \\ n-1 \end{smallmatrix}, \begin{smallmatrix} & n & \\ n-1 & & n-1 \end{smallmatrix} \right)$$

For a given $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$ and any $i \in I$, the brackets in the i-signature $S_i^c(\boldsymbol{\alpha})$ correspond to the stacks for the roots in Φ_i^B or Φ_i^C. As such, the crystal operators on Kostant partitions given in Definition II.30 act by adding or removing an i from the top of a stack and considering roots corresponding to the resultant stack. That is, f_i add i to the top of the stack corresponding to the leftmost '(' in $S_i^c(\boldsymbol{\alpha})$. This heuristic explanation of f_i is correct except the case of f_n for crystals of type C_n. In this situation, f_n adds or removes an n, and then either splits or combines two stacks in correspondence with the rules laid out in II.30.

To show the utility of this stack notation we now review some of our previous examples in type C_n. The first is an example of where f_n adds an n to a stack, and the second shows the case where f_n causes a stack to "split" into two positive roots.

Example IV.1. Let $\mathrm{Kp}(\infty)$ be of type C_3 and consider $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$ defined as in Example II.32. The Kostant partition can then be written in stack notation to give,

$$\boldsymbol{\alpha} = \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}\ 3\,.$$

The corresponding 3-signature is again,

$$\begin{array}{lccccccccccccc}
 & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}2\\1\end{smallmatrix} & \begin{smallmatrix}2\\1\end{smallmatrix} & \begin{smallmatrix}2\\1\end{smallmatrix} & \begin{smallmatrix}3\\2\\1\end{smallmatrix} & \begin{smallmatrix}3\\2\\1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix} & \begin{smallmatrix}3\\22\end{smallmatrix} & \begin{smallmatrix}3\\2\end{smallmatrix} & \begin{smallmatrix}3\\22\end{smallmatrix} & 3 \\
S_3(\boldsymbol{\alpha}) = &) &) & (& \color{red}(& \color{red}(& \color{red}) & \color{red}) & \color{red}(& \color{red}(& \color{red}) & \color{red}) & \color{red}(& \color{red}) \\
S_3^c(\boldsymbol{\alpha}) = &) &) & \color{blue}(& & & & & & & & & &
\end{array}.$$

Thus the action of f_3 on $\boldsymbol{\alpha}$ adds a 3 to top of a $\begin{smallmatrix}2\\1\end{smallmatrix}$ stack, equivalent to removing a $\beta_{1,2}$ and adding a $\gamma_{1,3}$. In stack notation this gives the result of,

$$f_3\boldsymbol{\alpha} = \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}\ 3\,.$$

Example IV.2. Let $\mathrm{Kp}(\infty)$ be of types C_3 and consider $\boldsymbol{\alpha} \in \mathrm{Kp}(\infty)$ defined as in Example II.34. Rewriting the Kostant partition $\boldsymbol{\alpha}$ by way of stack notation results in

$$\boldsymbol{\alpha} = \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\\1&&1\end{smallmatrix}\ \begin{smallmatrix}&3&\\2&&2\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}\ 3\,.$$

Recalculating the 3-signature using stack notation gives,

$$\begin{array}{rcccccccccccccc}
 & \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix} & \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix} & \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix} & \begin{smallmatrix}2\\1\end{smallmatrix} & \begin{smallmatrix}2\\1\end{smallmatrix} & \begin{smallmatrix}3\\2\\1\end{smallmatrix} & \begin{smallmatrix}3\\2\\1\end{smallmatrix} & \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix} & \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix} & \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix} & \begin{smallmatrix}3\\2\,2\end{smallmatrix} & \begin{smallmatrix}3\\2\end{smallmatrix} & \begin{smallmatrix}3\\2\,2\end{smallmatrix} & 3 \\
S_3(\boldsymbol{\alpha}) = &) &) &) & \color{red}{(} & \color{red}{(} & \color{red}{)} & \color{red}{)} & (& \color{red}{(} & \color{red}{(} & \color{red}{)} & \color{red}{)} & \color{red}{(} & \color{red}{)} \\
S_3^c(\boldsymbol{\alpha}) = &) &) &) & & & & & \color{blue}{(} & & & & & & \quad .
\end{array}$$

So, following the rules for the crystal operators given in II.30, Since the leftmost '(' of $S_3^c(\boldsymbol{\alpha})$ comes from a $\gamma_{1,1}$, the action of f_3 on $\boldsymbol{\alpha}$ would add a 3 to the stack, but this does not result in a positive root, so f_3 instead "splits" the $\gamma_{1,1}$ stack into two $\gamma_{1,3}$ stacks, as shown below:

$$f_3\boldsymbol{\alpha} = \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix}\ \begin{smallmatrix}3\\2\,2\\1\,1\end{smallmatrix}\ \begin{smallmatrix}3\\2\,2\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}\ 3\ .$$

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