THE WEIGHT FUNCTION FOR NAKAJIMA MONOMIALS OF TYPES $A_n^{(1)}$ AND $B_3^{(1)}$

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ABSTRACT

THE WEIGHT FUNCTION FOR NAKAJIMA MONOMIALS OF TYPES $A_n^{(1)}$ AND $B_3^{(1)}$

by Luke James

There exists a certain realization of the crystal $B(\infty)$ constructed using modified Nakajima monomials. The current model is incomplete for affine crystals as the weight function fails to account for the null root δ. This thesis accounts for the coefficient of δ in the weight function for monomial crystals of type $A_n^{(1)}$ and $B_3^{(1)}$, then discusses some difficulties in accounting for the δ coefficient in type $B_4^{(1)}$.

TABLE OF CONTENTS

LIST OF FIGURES vi

CHAPTER

I. BACKGROUND INFORMATION 1
I.1. Lie Algebras 1
I.2. Affine Kac-Moody Algebras 1
I.3. Affine Crystals 4
I.3.1. Affine Quantum Groups 4
I.3.2. Affine Crystals 5
I.3.3. $B(\infty)$ 6
I.4. Generalized Young Walls 7
I.5. Modified Nakajima Monomials 11
I.5.1. Defining the Modified Nakajima Monomials 11
I.5.2. The Weight Function of the Modified Nakajima Monomials 14

II. TYPE A WEIGHT FUNCTION 16
II.1. The Result for Type $A_1^{(1)}$ 16
II.2. The Result for Type $A_n^{(1)}$ 22
II.2.1. Examples 26

III. TYPE B WEIGHT FUNCTION 28
III.1. The Result for $B_3^{(1)}$ 28
III.2. An Attempt at $B_4^{(1)}$ 34

REFERENCES 38

LIST OF FIGURES

FIGURE PAGE

1. The top part of the crystal $\mathscr{Y}(\infty)$ in type $A_2^{(1)}$, created using SAGEMATH [7]. 10

2. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_2^{(1)}$ in terms of A-variables, created using SAGEMATH [7]. 14

3. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_2^{(1)}$ in terms of Y-variables, created using SAGEMATH [7]. 15

4. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_1^{(1)}$ in terms of the Y-variables, created using SAGEMATH [7]. 17

5. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_1^{(1)}$ in terms of the A-variables, created using SAGEMATH [7]. 18

6. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $B_3^{(1)}$ in terms of the Y-variables, created using SAGEMATH [7]. 36

7. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $B_3^{(1)}$ in terms of the A-variables, created using SAGEMATH [7]. 37

CHAPTER I

BACKGROUND INFORMATION

I.1. Lie Algebras

A vector space L over $\mathbb{C}$ with a bilinear operation $[\ ,\ ]\colon L \times L \to L$ called the **(Lie) bracket**, is a **Lie algebra** if the bracket satisfies

1. $[x,x] = 0$ for all $x \in L$,
2. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in L$.

Furthermore, given a Lie algebra L and any $x \in L$, define the **adjoint map** $\mathrm{ad}(x)\colon L \to L$ by

$$\mathrm{ad}(x)(y) = [x,y],$$

for all $y \in L$. This map will be important for defining Kac-Moody algebras later.

I.2. Affine Kac-Moody Algebras

For this section, we follow [1, Chapters 2 and 10]. For more information, see [2].

Let I be a finite index set. A square matrix $C = (C_{ij})_{i,j \in I}$ is called a **generalized Cartan matrix** if

- $C_{ij} \in \mathbb{Z}$ for all $i, j \in I$;
- $C_{ii} = 2$ for all $i \in I$;
- $C_{ij} \leq 0$ if $i \neq j$;
- $C_{ij} = 0$ if and only if $C_{ji} = 0$.

Furthermore, if there exists a diagonal matrix $D = \mathrm{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that DC is symmetric, then C is said to be **symmetrizable**. A generalized Cartan matrix C is said to be of **affine type** if the corank of C is 1, there exists a $u \in \mathbb{R}^{|I|}$ such that each component of u is positive and $Cu = 0$, and such that the components of Cv are all nonnegative if and only if $Cv = 0$ for all $v \in \mathbb{R}^{|I|}$.

Remark I.2.1. In the theory of semisimple Lie algebras, Cartan matrices arise from the construction of the corresponding root system. Generalized Cartan matrices are an axiomatization of the properties of the Cartan matrices.

Let $I = \{0, \ldots, n\}$ be an index set and let $C = (C_{ij})_{i,j \in I}$ be a symmetrizable Cartan matrix of affine type. Let $P^\vee$ be a free abelian group of rank $n+2$ with $\mathbb{Z}$-basis $\{h_i \mid i \in I\} \cup \{d\}$ and let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ be the complexification of $P^\vee$. Call $P^\vee$ the **dual weight lattice**.

Now define linear functionals α_i and Λ_i ($i \in I$) on $\mathfrak{h}$ by

$$\alpha_i(h_j) = C_{ji}, \qquad \alpha_i(d) = \delta_{0,i},$$
$$\Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(d) = 0 \; (i, j \in I).$$

The α_i are called the **simple roots**, the h_i are called the **simple coroots**, and the Λ_i are called the **fundamental weights**. Denote $\Pi = \{\alpha_i \mid i \in I\}$ and $\Pi^\vee = \{h_i \mid i \in I\}$. Finally, define the **weight lattice** to be

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}.$$

Definition I.2.2. The quintuple $(C, \Pi, \Pi^\vee, P, P^\vee)$ forms an **affine Cartan datum** associated with the symmetrizable affine generalized Cartan matrix $C = (C_{ij})_{i,j \in I}$.

Note that since P and $P^\vee$ are dual to each other, there is an associated canonical pairing $\langle \, , \, \rangle : P^\vee \times P \to \mathbb{Z}$ given by $\langle h, \lambda \rangle = \lambda(h)$ for all $h \in P^\vee, \lambda \in P$.

Definition I.2.3. Let $(C, \Pi, \Pi^\vee, P, P^\vee)$ be an affine Cartan datum. The **affine Kac-Moody algebra** $\mathfrak{g}$ associated with this affine Cartan datum is the Lie algebra generated by the elements e_i, f_i ($i \in I$) and $h \in P^\vee$ subject to the following relations:

1. $[h, h'] = 0$ for $h, h' \in P^\vee$,
2. $[e_i, f_j] = \delta_{ij} h_i$,
3. $[h, e_i] = \alpha_i(h) e_i$ for $h \in P^\vee$,
4. $[h, f_i] = -\alpha_i(h) f_i$ for $h \in P^\vee$,

5. $(\mathrm{ad}\, e_i)^{1-C_{ij}} e_j = 0$ for $i \neq j$,
6. $(\mathrm{ad}\, f_i)^{1-C_{ij}} f_j = 0$ for $i \neq j$.

An affine Kac-Moody algebra has a **null root**

$$\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_n\alpha_n$$

for certain nonnegative integers d_i $(i \in I)$ (see [2, Table Aff] for more information). Using the null root, the affine weight lattice can be expressed as

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\frac{1}{d_0}\delta.$$

Furthermore, each affine Kac-Moody algebra has a unique element θ, which is expressible as an element of $\bigoplus_{i=0}^n \mathbb{Z}\Lambda_i$, such that $\delta = d_0\alpha_0 + \theta$ for some $d_0 \in \mathbb{Z}$. For types $A_n^{(1)}$ and $B_n^{(1)}$ (which will be defined shortly), $d_0 = 1$. Moreover, for an affine Kac-Moody algebra, the simple root α_j, for $j \neq 0$, is always expressible as an element of $\bigoplus_{i=0}^n \mathbb{Z}\Lambda_i$.

Example I.2.4. The Kac-Moody algebra of type $A_1^{(1)}$ is the affine Kac-Moody algebra with Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

In this case, $\delta = \alpha_0 + \alpha_1$.

Example I.2.5. For $n \geq 2$, the Kac-Moody algebra of type $A_n^{(1)}$ is the affine Kac-Moody algebra with Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Here, $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n$.

Example I.2.6. For $n \geq 3$, the Kac-Moody algebra of type $B_n^{(1)}$ is the affine Kac-Moody algebra with Cartan matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}$$

and has $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + 2\alpha_n$.

I.3. Affine Crystals

I.3.1. Affine Quantum Groups

In this section and the next, we follow the exposition of [1, Chapters 3 and 10].

Given any $n \in \mathbb{Z}$ and any indeterminate q, define the **q-integers**

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Define $[0]_q! = 1$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ for all $n \in \mathbb{Z}_{>0}$. Using these values, define the **q-binomial coefficients** to be

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}.$$

Suppose that C is an affine symmetrizable generalized Cartan matrix with diagonal matrix $D = \mathrm{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that DC is symmetric.

Definition I.3.1. Let $\mathfrak{g}$ be a Kac-Moody algebra associated with a Cartan datum $(C, \Pi, \Pi^\vee, P, P^\vee)$. The **affine quantum group** $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$ is the associative algebra over $\mathbb{C}(q)$ with 1 generated by the elements e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ with the following defining relations:

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$,
4. $e_i f_i - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,

5. $\sum_{k=0}^{1-C_{ij}} (-1)^k \begin{bmatrix} 1-C_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-C_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,
6. $\sum_{k=0}^{1-C_{ij}} (-1)^k \begin{bmatrix} 1-C_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-C_{ij}-k} f_j f_i^k$ for $i \neq j$.

Here, $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$.

The quantum group $U_q(\mathfrak{g})$ is a q-analogue to the universal enveloping algebra of $\mathfrak{g}$ (for more information, see [1]).

Let $U_q^+(\mathfrak{g})$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by e_i ($i \in I$) and $U_q^-(\mathfrak{g})$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by f_i ($i \in I$). Furthermore, let $U_q^0(\mathfrak{g})$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by the q^h ($h \in P^\vee$). Then $U_q(\mathfrak{g})$ has **triangular decomposition**

$$U_q(\mathfrak{g}) \cong U_q^+(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^-(\mathfrak{g}).$$

where the isomorphism is as $\mathbb{C}(q)$-vector spaces.

I.3.2. Affine Crystals

Definition I.3.2. An **affine crystal** for the affine quantum group $U_q(\mathfrak{g})$ (or a $U_q(\mathfrak{g})$**-crystal**) is a set B together with the maps $\mathrm{wt}\colon B \to P$, $\varepsilon_i, \varphi_i\colon B \to \mathbb{Z} \cup \{-\infty\}$, $\tilde{e}_i, \tilde{f}_i\colon B \to B \cup \{0\}$ ($i \in I$) such that for all $i \in I$ and $b \in B$,

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$,
2. $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$,
3. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi(\tilde{e}_i b) = \varphi_i(b) + 1$,
4. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$,
5. $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$ for $b, b' \in B$,
6. $\tilde{e}_i b = \tilde{f}_i b = 0$ if $\varepsilon_i(b) = -\infty$.

The operators $\tilde{e}_i$ and $\tilde{f}_i$ above are known as the **Kashiwara operators**. Note that condition 5 means that the Kashiwara operators define an I-colored directed graph structure on B given by $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i b = b'$. (Note that $\tilde{f}_i b$ may be zero, in which case $\tilde{e}_i \tilde{f}_i b \neq b$; there are no arrows of the form $b \xrightarrow{i} 0$ because 0 is not considered to be an element of B.) This graph is called the **crystal graph** of B. Examples of these graphs will be given later.

Definition I.3.3. A **morphism** $\psi\colon B_1 \to B_2$ of crystals is a map $\psi\colon B_1 \cup \{0\} \to B_2 \cup \{0\}$ satisfying the following conditions:

1. $\psi(0) = 0$,
2. $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $b \in B_1$ such that $\psi(b) \neq 0$,
3. $\tilde{f}_i\psi(b) = \psi(b')$ if $b, b' \in B_1$ and $\tilde{f}_i b = b'$.

From this, an **isomorphism** of crystals can be defined as a bijective morphism of crystals such that $\psi(\tilde{f}_i b) = \tilde{f}_i\psi(b)$ for all $b \in B_1$ and $i \in I$.

I.3.3. $B(\infty)$

Let $\tilde{e}_i$, $\tilde{f}_i$ be the Kashiwara operators on $U_q^-(\mathfrak{g})$ defined in [4]. We will not require the algebraic definition of the Kashiwara operators, so their explicit definition is omitted. Let $\mathscr{A} \subset \mathbb{C}(q)$ be the subring of functions regular at $q = 0$; i.e., $\mathscr{A} = \left\{ \frac{g(q)}{h(q)} : g, h \in \mathbb{C}[q], h(0) \neq 0 \right\}$. Define $L(\infty)$ to be the $\mathscr{A}$-lattice spanned by

$$S = \{\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_t}1 \in U_q^-(\mathfrak{g}) : t \geq 0,\ i_k \in I\}.$$

Theorem/Definition I.3.4 ([4])**.**

1. Let $\pi\colon L(\infty) \to L(\infty)/qL(\infty)$ be the natural projection and set $B(\infty) = \pi(S)$. Then $B(\infty)$ is a $\mathbb{C}$-basis of $L(\infty)/qL(\infty)$.

2. For each $i \in I$ the operators $\tilde{e}_i$ and $\tilde{f}_i$ act on $L(\infty)/qL(\infty)$. Moreover, $\tilde{e}_i\big(B(\infty)\big) = B(\infty) \sqcup \{0\}$ and $\tilde{f}_i\big(B(\infty)\big) \subset B(\infty)$.

The **highest weight vector** in $B(\infty)$ is denoted u_∞ and satisfies $\tilde{e}_i u_\infty = 0$, for all $i \in I$, and $\mathrm{wt}(u_\infty) = 0$. The element u_∞ is $\pi(1)$. For $i \in I$ and $b \in B(\infty)$, define the maps

$$\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} : \tilde{e}_i^k b \neq 0\},$$

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle,$$

$$\mathrm{wt}(\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_t}u_\infty) = -\alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_t}.$$

Here, recall that $\langle\ ,\ \rangle$ is the canonical pairing between $P^\vee$ and P. Then $B(\infty)$, along with the maps defined above forms a $U_q(\mathfrak{g})$-crystal.

I.4. Generalized Young Walls

In this section, we recall the results from [5].

As before, set $I = \{0,1,\ldots,n\}$, but we understand these indices as elements of $\mathbb{Z}/(n+1)\mathbb{Z}$. Let $\mathscr{B}$ be a board with coloring as follows:

	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\cdots$	0	1	2	$\cdots$	0	1
$\cdots$	n	0	1	$\cdots$	n	0
$\cdots$	$n-1$	n	0	$\cdots$	$n-1$	n
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\cdots$	0	1	2	$\cdots$	0	1
$\cdots$	n	0	1	$\cdots$	n	0

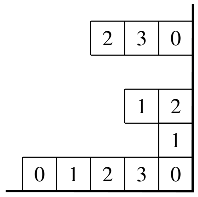
.

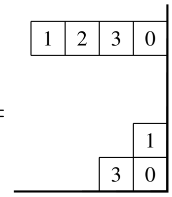
Definition I.4.1. The **generalized Young walls** are constructed of I-colored boxes on the board $\mathscr{B}$ subject to the conditions:

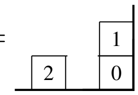
1. the boxes are colored according to the board;
2. the colored boxes are placed in rows starting from the right.

Definition I.4.2. A generalized Young wall is said to be **proper** if, for each $p > q$ such that $p - q \equiv 0 \bmod n+1$, the number of boxes in the pth row from the bottom is less than or equal to the number of boxes in the qth row from the bottom.

Example I.4.3. Consider the following arrangement of boxes on the board $\mathscr{B}$ for $n = 3$.

$Y =$ 

, $Y' =$ 

, $Y'' =$ 

.

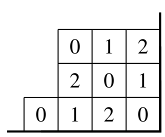
Then Y and Y' are both generalized Young walls, but Y'' is not since there is a break in the first row. Furthermore, Y is proper. The wall Y' is not proper because the fifth row has four elements but the first row has two elements, and $5 - 1 \equiv 0 \bmod 4$.

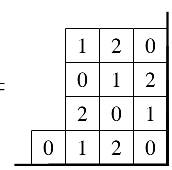
Definition I.4.4. The kth column Y_k of a generalized Young wall contains a **removable** δ if one of each i colored box can be removed from Y_k and still obtain a generalized Young wall. In other words, if $a_{i,k}$ is the number of i colored boxes in the kth column Y_k, then Y_k contains a removable δ if

$$a_{i-1,k+1} < a_{i,k} \qquad \text{for all } i \in I.$$

If a generalized Young wall contains no removable δ, it is said to be **reduced**.

Example I.4.5. Consider the following generalized Young walls for $n = 2$.

$Y =$ 

, $Y' =$ 

.

Then Y is reduced since removing a $0, 1$, and 2 from any given column would leave the 0 in the fourth column separated from its row, and therefore no δ is removable. On the other hand, Y' has a removable δ in the third column, so is not reduced.

Let $\mathscr{F}(\infty)$ denote the set of all proper generalized Young walls. Let $\mathscr{Y}(\infty)$ denote the set of all proper reduced generalized Young walls. Note that $\mathscr{Y}(\infty) \subset \mathscr{F}(\infty)$.

Given any $Y \in \mathscr{F}(\infty)$, say that the leftmost box of any row is **removable** and, if it is i-colored, then it is called a **removable i-box**. Also, define the site left of the leftmost box in each row to be **admissible**, and if a row has no boxes, then its first site is admissible. If the site is i-colored, then it is called an ***i*-admissible slot**.

For any $Y \in \mathscr{F}(\infty)$, let $y_1, y_2, \ldots$ be the removable i-boxes and i-admissible slots ordered from left to right and bottom to top. The i-signature of y_j is said to be $-$ if y_j is removable and $+$ if y_j is admissible. Then the i-signature of Y is obtained by producing the sequence of i-signatures of $y_1, y_2, \ldots$ and then canceling out any $(+,-)$ pairs, resulting in a sequence of $-$'s followed by $+$'s. An example is given after defining the crystal operators on $\mathscr{F}(\infty)$ below.

Define $\tilde{f}_i Y$ to be the proper generalized Young wall obtained by placing an i-colored box at the site corresponding to the leftmost $+$ in the i-signature of Y and $\tilde{e}_i Y$ to be the proper generalized Young wall obtained by removing the i-box corresponding to the rightmost $-$ in the i-signature of Y. If no such $-$ exists, define $\tilde{e}_i Y = 0$. Also, define the maps

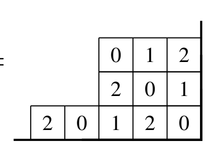
$$\begin{aligned} \mathrm{wt}(Y) &= -\sum_{i \in I} k_i \alpha_i, \\ \varepsilon_i(Y) &= \text{ the number of } -\text{'s in the } i\text{-signature of } Y, \\ \varphi_i(Y) &= \varepsilon_i(Y) + \langle h_i, \mathrm{wt}(Y) \rangle. \end{aligned}$$

Here, k_i is the number of i colored boxes in Y and the α_i are as defined for $U_q(A_n^{(1)})$. Then $\mathscr{F}(\infty)$ together with the maps above form a $U_q(A_n^{(1)})$-crystal. For $\mathscr{Y}(\infty)$, more can be said.

Theorem I.4.6 ([5]). *The morphism $B(\infty) \to \mathscr{Y}(\infty)$ such that $u_\infty \mapsto \emptyset$, where $\emptyset$ is the empty generalized Young wall, defines a $U_q(A_n^{(1)})$-crystal isomorphism.*

The first three levels of the crystal graph of $\mathscr{Y}(\infty)$ can be seen in Figure 1 along with the SAGEMATH code used to generate it.

Example I.4.7. Consider $Y \in \mathscr{Y}(\infty)$ with $n = 2$ such that

$Y =$ 

.

```
sage: Y = crystals.infinity.GeneralizedYoungWalls(2)
sage: S = Y.subcrystal(max_depth=2)
sage: G = Y.digraph(subset=S)
sage: latex(G)
```

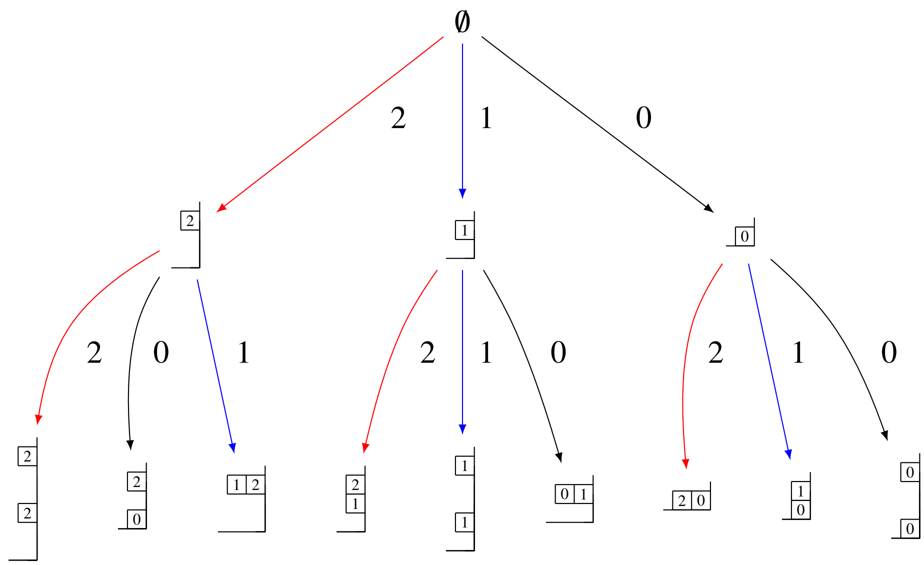
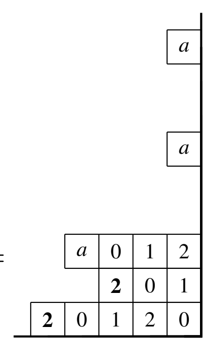


Figure 1. The top part of the crystal $\mathscr{Y}(\infty)$ in type $A_2^{(1)}$, created using SAGEMATH [7].

To calculate $\tilde{f}_2 Y$, consider the following 2-admissible slots (denoted by a) and removable 2-blocks (bolded for emphasis).

$Y =$  .

Thus, the 2-signatures of 2-admissible slots and removable 2-blocks is

$$-,+,-,+,+,\cdots,$$

so the 2-signature of Y is

$$-,+,+,+,\cdots.$$

The leftmost $+$ of the 2-signature originally corresponds to the 2-admissible slot in the sixth row, therefore

$$\tilde{f}_2 Y = \begin{array}{|c|c|c|c|c|} \hline & & & & 2 \\ & & & & \\ & & & & \\ & & 0 & 1 & 2 \\ & & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 & 0 \\ \hline \end{array} .$$

Similarly, the rightmost $-$ of the 2-signature corresponds to the removable 2-block in the first row, therefore

$$\tilde{e}_2 Y = \begin{array}{|c|c|c|c|} \hline & 0 & 1 & 2 \\ & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ \hline \end{array} .$$

To compute the weight, note that Y has four 0-boxes, three 1-boxes, and four 2-boxes. Thus, $\mathrm{wt}(Y) = -4\alpha_0 - 3\alpha_1 - 4\alpha_2$. The 2-signature of Y is $-,+,+,+,\cdots$ so $\varepsilon_2(Y) = 1$. Finally, this means that

$$\begin{aligned} \varphi_2(Y) &= \varepsilon_2(Y) + \langle h_2, \mathrm{wt}(Y) \rangle \\ &= 1 + -4\alpha_0(h_2) - 3\alpha_1(h_2) - 4\alpha_2(h_2) \\ &= 1 - 4(0) - 3(-1) - 4(2) \\ &= -4. \end{aligned}$$

I.5. Modified Nakajima Monomials

In this section, we recall the results from [3].

I.5.1. Defining the Modified Nakajima Monomials

Let $(C, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum. Let $Y_{i,k}$ $(i \in I, k \in \mathbb{Z})$ be formal commuting variables with an additional commuting variable $\mathbf{1}$. Define the **modified Nakajima monomials** with respect to this Cartan datum to be the set $\widehat{\mathscr{M}}$ of all monomials of the form

$$M = \prod_{i \in I} \prod_{k \in \mathbb{Z}_{\geq 0}} Y_{i,k}^{y_{i,k}} \mathbf{1},$$

where $y_{i,k} \in \mathbb{Z}$ and $y_{i,k} = 0$ for all but finitely many k. For such an M, define

$$\begin{aligned}
\mathrm{wt}(M) &= \sum_{i\in I}\left(\sum_{k\in\mathbb{Z}_{\geq 0}} y_{i,k}\right)\Lambda_i,\\
\varphi_i(M) &= \max\left\{\sum_{j=0}^{k} y_{i,j} \mid k\in\mathbb{Z}_{\geq 0}\right\},\\
\varepsilon_i(M) &= \varphi(M) - \langle h_i, \mathrm{wt}(M)\rangle,\\
k_f = k_f(M) &= \min\left\{k\in\mathbb{Z}_{\geq 0} \mid \varphi_i(M) = \sum_{j=0}^{k} y_{i,j}\right\},\\
k_e = k_e(M) &= \max\left\{k\in\mathbb{Z}_{\geq 0} \mid \varphi_i(M) = \sum_{j=0}^{k} y_{i,j}\right\}.
\end{aligned}$$

Next, choose a set of nonnegative integers $(o_{ij})_{i\neq j}$ such that $o_{ij} + o_{ji} = 1$. Define

$$A_{i,k} = Y_{i,k}Y_{i,k+1}\prod_{j\neq i} Y_{j,k+o_{ji}}^{C_{ji}}.$$

Then the Kashiwara operators can be defined as

$$\tilde{e}_i M = \begin{cases} 0 & \text{if } \varepsilon_i(M) = 0,\\ A_{i,k_e}M & \text{if } \varepsilon_i(M) > 0, \end{cases} \qquad \tilde{f}_i M = A_{i,k_f}^{-1}M.$$

The set $\widehat{\mathscr{M}}$ together with the maps $\mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i$ $(i\in I)$ forms a $U_q(\mathfrak{g})$-crystal [3].

Example I.5.1. Let $\mathfrak{g}$ be the Kac-Moody algebra of type $A_2^{(1)}$. Let $\widehat{\mathscr{M}}$ be as defined above, with $(o_{ij})_{i\neq j}$ given by

$$\begin{pmatrix} & 1 & 0\\ 0 & & 1\\ 1 & 0 & \end{pmatrix}.$$

Recall that the Cartan matrix of $\mathfrak{g}$ is given by

$$\begin{pmatrix} 2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2 \end{pmatrix}.$$

Using these values, it can be computed that

$$A_{i,k} = Y_{i,k}Y_{i,k+1}Y_{i-1,k+1}^{-1}Y_{i+1,k}^{-1}$$

where addition in I is given by addition in $\mathbb{Z}/(n+1)\mathbb{Z}$.

Let $M = \tilde{f}_0\mathbf{1} = A_{0,0}^{-1}\mathbf{1} = Y_{0,0}^{-1}Y_{1,0}Y_{0,1}^{-1}Y_{2,1}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$. Note that

$$\varphi_2(M) = \max\left\{\sum_{j=0}^{k} y_{2,j} \mid k \in \mathbb{Z}_{\geq 0}\right\} = \max\{0,1,1,\ldots\} = 1.$$

Using this, it can be seen that

$$k_f(M) = \min\left\{k \in \mathbb{Z}_{\geq 0} \mid \varphi_i(M) = \sum_{j=0}^{k} y_{2,j}\right\} = 1.$$

Therefore, $\tilde{f}_2 M$ is given by

$$\begin{aligned}\tilde{f}_2 M &= A_{2,1}^{-1} M \\ &= Y_{2,1}^{-1}Y_{2,2}^{-1}Y_{1,2}Y_{0,1}Y_{0,0}^{-1}Y_{1,0}Y_{0,1}^{-1}Y_{2,1} \\ &= Y_{0,0}^{-1}Y_{1,0}Y_{1,2}Y_{2,2}^{-1}.\end{aligned}$$

Now define $\widehat{\mathscr{M}}(\mathbf{1})$ to be the connected component of $\widehat{\mathscr{M}}$ (under the application of the Kashiwara operators) containing $\mathbf{1}$.

Theorem I.5.2 ([3]). *The morphism $B(\infty) \to \widehat{\mathscr{M}}(\mathbf{1})$ given by $u_\infty \mapsto \mathbf{1}$ is a $U_q(\mathfrak{g})$-crystal isomorphism.*

It can be seen using results from [5] that $\mathscr{Y}(\infty)$ is isomorphic as a crystal to $\widehat{\mathscr{M}}(\mathbf{1})$ under the map that sends a proper reduced generalized Young wall Y to $\prod_{i\in I}\prod_{k\geq 0} A_{i,k}^{-a_{i,k}}\mathbf{1}$ where $a_{i,k}$ is the number of i-colored boxes in the kth column of Y. This isomorphism can be observed by comparing the crystal graphs in Figure 1 and Figure 2. Figure 3 is the same crystal graph as seen in Figure 2 but with the $A_{i,k}$ expanded in terms of the $Y_{i,k}$.

```
sage: c = Matrix([[0,1,0],[0,0,1],[1,0,0]])
sage: M = crystals.infinity.NakajimaMonomials(['A',2,1], c=c)
sage: M.set_variables("A")
sage: S = M.subcrystal(max_depth=2)
sage: G = M.digraph(subset=S)
sage: latex(G)
```

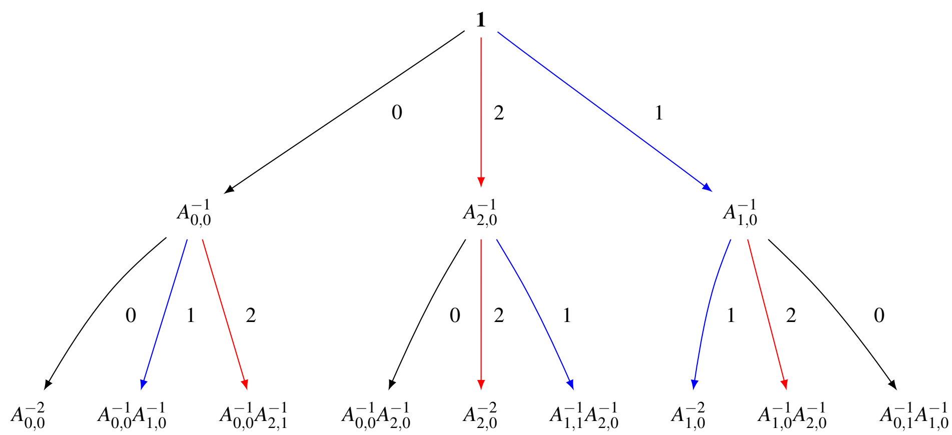


Figure 2. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_2^{(1)}$ in terms of A-variables, created using SAGEMATH [7].

I.5.2. The Weight Function of the Modified Nakajima Monomials

The weight function on the modified Nakajima monomials as given above is not complete for affine crystals.

Example I.5.3. Consider the $U_q(A_1^{(1)})$-crystal $\widehat{\mathscr{M}}(\mathbf{1})$ and set $M = \tilde{f}_1\tilde{f}_0\mathbf{1}$. Using the crystal axioms, it can be seen that $\mathrm{wt}(M) = -\alpha_0 - \alpha_1$. Since $\delta = \alpha_0 + \alpha_1$, this means that $\mathrm{wt}(M) = -\delta$. However, using the definition of the Kashiwara operators on $\widehat{\mathscr{M}}(\mathbf{1})$, note that $\tilde{f}_1\tilde{f}_0\mathbf{1} = Y_{0,0}^{-1}Y_{0,1}Y_{1,1}Y_{1,2}^{-1}\mathbf{1}$, so using the definition of the weight given in [3] gives $\mathrm{wt}(M) = 0$.

Note that in general, for any $M \in \widehat{\mathscr{M}}(\mathbf{1})$,

$$\mathrm{wt}(\tilde{f}_0 M) = \mathrm{wt}(M) - \alpha_0 = \mathrm{wt}(M) + \theta - \delta.$$

```
sage: c = Matrix([[0,1,0],[0,0,1],[1,0,0]])
sage: M = crystals.infinity.NakajimaMonomials(['A',2,1], c=c)
sage: S = M.subcrystal(max_depth=2)
sage: G = M.digraph(subset=S)
sage: latex(G)
```

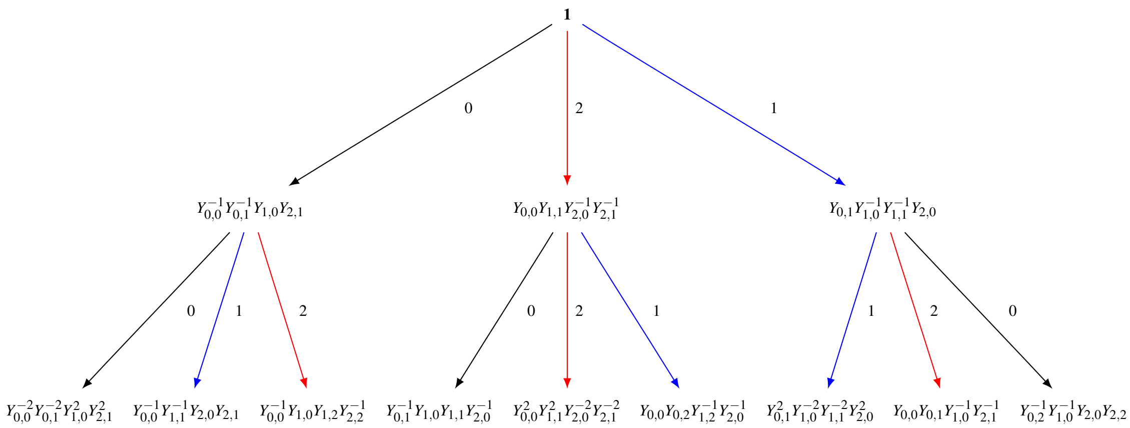


Figure 3. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_2^{(1)}$ in terms of Y-variables, created using SAGEMATH [7].

As a matter of fact, the original weight function given in [3] is actually correct when considered as a map wt: $\widehat{\mathscr{M}}(\mathbf{1}) \to P/\delta\mathbb{Z}$. Thus, the only missing value in the weight function is the coefficient on δ.

Since each of θ and α_i $(i \in I \setminus \{0\})$ can be expressed as an element of $\bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i$, it must be that, for $M = \tilde{f}_{b_1}\tilde{f}_{b_2}\cdots\tilde{f}_{b_\ell}\mathbf{1} = \prod_{i\in I}\prod_{k\geq 0} Y_{i,k}^{y_{i,k}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$, that

$$\mathrm{wt}(M) = \sum_{i\in I}\left(\sum_{k\geq 0} y_{i,k}\right)\Lambda_i + |\{1 \leq j \leq \ell \mid b_j = 0\}|\delta.$$

Note that this implies that, whenever M can be expressed uniquely as $\prod_{i\in I}\prod_{k\geq 0} A_{i,k}^{a_{i,k}}\mathbf{1}$, the δ coefficient is exactly $\sum_{k\geq 0} a_{0,k}$. Therefore, to complete the weight function for affine crystals, it suffices to calculate the number of 0-arrows applied to reach M in the crystal graph.

CHAPTER II

TYPE A WEIGHT FUNCTION

II.1. The Result for Type $A_1^{(1)}$

This section will focus on finding the full weight function for the $U_q(A_1^{(1)})$-crystal $\widehat{\mathscr{M}}(\mathbf{1})$. Set $o_{10} = 1$ and $o_{01} = 0$. Here, the $A_{i,k}$ are

$$A_{0,k} = Y_{0,k}Y_{0,k+1}Y_{1,k+1}^{-2},$$
$$A_{1,k} = Y_{1,k}Y_{1,k+1}Y_{0,k}^{-2}.$$

The first four levels of the crystal graph of $\widehat{\mathscr{M}}(\mathbf{1})$ in this case in terms of the Y-variables is given in Figure 4, and the graph in terms of the A-variables in given in Figure 5.

Lemma II.1.1. *For* $M = \prod_{k\in\mathbb{Z}_{\geq 0}} Y_{0,k}^{y_{0,k}} Y_{1,k}^{y_{1,k}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$*, define* $a_{i,k}$ *recursively as follows:*

$$a_{1,0} = y_{1,0},$$
$$a_{0,0} = y_{0,0} + 2a_{1,0},$$
$$a_{1,k} = y_{1,k} + 2a_{0,k-1} - a_{1,k-1},$$
$$a_{0,k} = y_{0,k} + 2a_{1,k} - a_{0,k-1}.$$

Then

$$M = \prod_{k\in\mathbb{Z}_{\geq 0}} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\mathbf{1}.$$

Proof. Since $M \in \widehat{\mathscr{M}}(\mathbf{1})$, there exists some $a_{0,k}, a_{1,k} \in \mathbb{Z}$ such that $M = \prod_{k\in\mathbb{Z}_{\geq 0}} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\mathbf{1}$. Thus, it suffices to show that the recurrence holds for these values $a_{i,k}$. Expanding the terms $A_{0,k}^{a_{0,k}}$ and $A_{1,k}^{a_{1,k}}$ shows that

$$M = Y_{0,0}^{a_{0,0}-2a_{1,0}} Y_{1,0}^{a_{1,0}} Y_{0,1}^{a_{0,1}} Y_{1,1}^{a_{1,1}-2a_{0,1}} \prod_{k\in\mathbb{Z}_{\geq 1}} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\mathbf{1}.$$

```
sage: c = Matrix([[0,0],[1,0]])
sage: M = crystals.infinity.NakajimaMonomials(['A',1,1], c=c)
sage: S = M.subcrystal(max_depth=3)
sage: G = M.digraph(subset=S)
sage: latex(G)
```

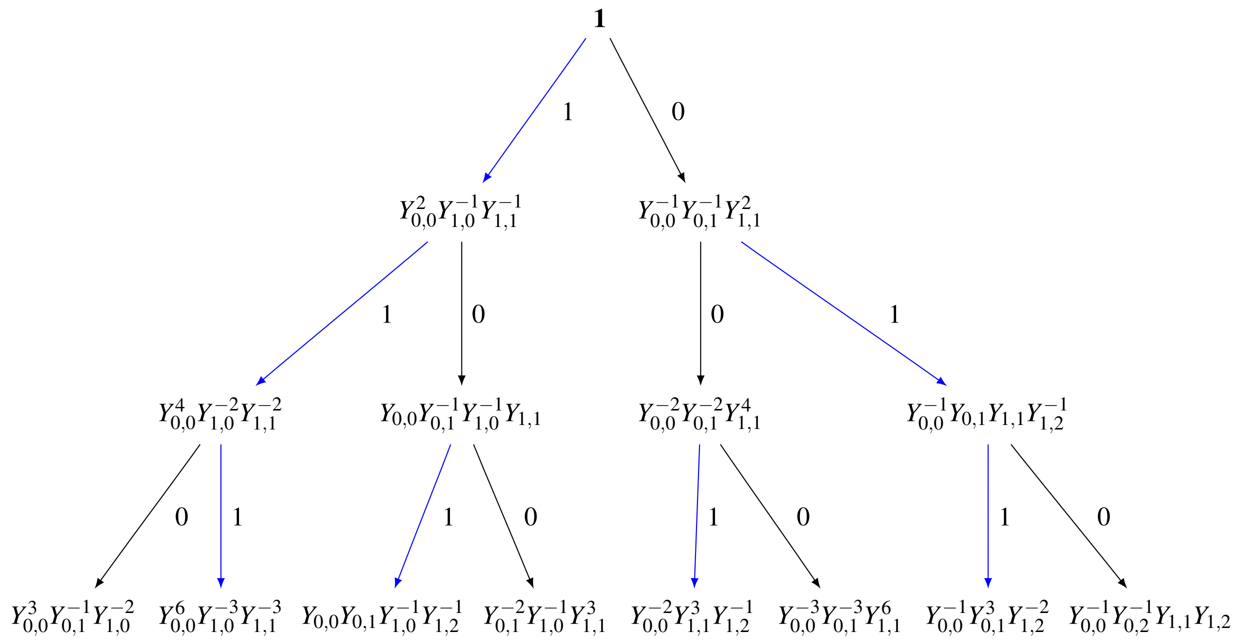


Figure 4. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_1^{(1)}$ in terms of the Y-variables, created using SAGEMATH [7].

Since none of the terms in $\prod_{k\in\mathbb{Z}_{\geq 1}} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\mathbf{1}$ contribute to the power on $Y_{0,0}$ or $Y_{1,0}$, equating powers yields $y_{1,0} = a_{1,0}$ and $y_{0,0} = a_{0,0} - 2a_{1,0}$, so $a_{0,0} = y_{0,0} + 2a_{1,0}$. Thus, the first two desired equations hold.

Next, for some $m \in \mathbb{Z}_{\geq 1}$, consider

$$M = \left(\prod_{k=0}^{m-2} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\right) A_{0,m-1}^{a_{0,m-1}} A_{1,m-1}^{a_{1,m-1}} A_{0,m}^{a_{0,m}} A_{1,m}^{a_{1,m}} \left(\prod_{k=m+1}^{\infty} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\right)\mathbf{1}.$$

Note that $\prod_{k=0}^{m-2} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\mathbf{1}$, when expanded, only yields values of $Y_{i,k}$ where $k < m$, and the product $\prod_{k=m+1}^{\infty} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}}\mathbf{1}$ will only yield powers of $Y_{i,k}$ where $k > m$. Thus, in the expansion of

```
sage: c = Matrix([[0,0],[1,0]])
sage: M = crystals.infinity.NakajimaMonomials(['A',1,1], c=c)
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sage: S = M.subcrystal(max_depth=3)
sage: G = M.digraph(subset=S)
sage: latex(G)
```

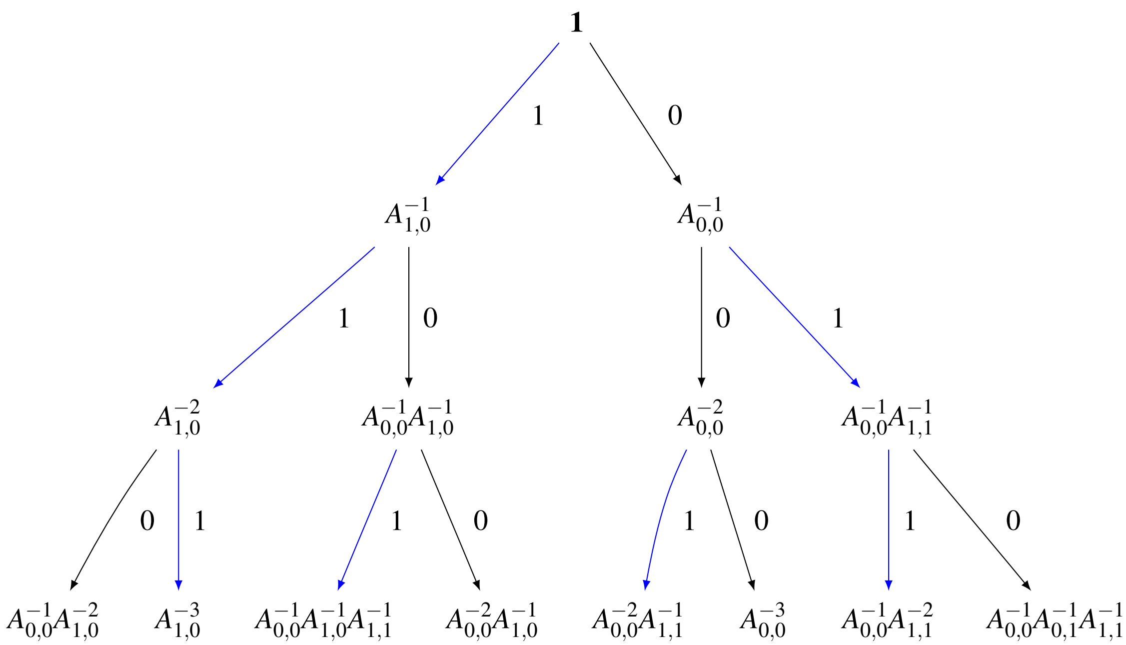


Figure 5. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $A_1^{(1)}$ in terms of the A-variables, created using SageMath [7].

$A_{1,m-1}^{a_{1,m-1}}A_{0,m}^{a_{0,m}}A_{1,m}^{a_{1,m}}$, the powers on $Y_{0,m}$ and $Y_{1,m}$ will be exactly $y_{0,m}$ and $y_{1,m}$ respectively. Since

$$A_{1,m-1}^{a_{1,m-1}}A_{0,m}^{a_{0,m}}A_{1,m}^{a_{1,m}} = Y_{0,m-1}^{a_{0,m-1}-2a_{1,m-1}}Y_{1,m-1}^{a_{1,m-1}}Y_{0,m}^{a_{0,m-1}-2a_{1,m}+a_{0,m}}Y_{1,m}^{a_{1,m-1}-2a_{0,m-1}+a_{1,m}},$$

equating powers gives $y_{0,m} = a_{0,m-1}-2a_{1,m}+a_{0,m}$ and $y_{1,m} = a_{1,m-1}-2a_{0,m-1}+a_{1,m}$. These can be rearranged to yield the remaining desired equations. □

For the following lemma, use the convention that an empty sum is 0.

Lemma II.1.2. *Let* $M = \prod_{k\in\mathbb{Z}_{\geq 0}} Y_{0,k}^{y_{0,k}} Y_{1,k}^{y_{1,k}} \mathbf{1} = \prod_{k\in\mathbb{Z}_{\geq 0}} A_{0,k}^{a_{0,k}} A_{1,k}^{a_{1,k}} \mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$. *Then*

$$a_{0,k} = \sum_{i=0}^{k} (2i+1) y_{0,k-i} + (2i+2) y_{1,k-i}$$

and

$$a_{1,k} = (2k+1) y_{1,0} + \sum_{i=0}^{k-1} (2i+1) y_{1,k-i} + (2i+2) y_{0,k-i-1}$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. Note that the equations of Lemma II.1.1 can be rewritten as a single recurrence relation. Namely, for $i \geq 1$, define $z_{2i-1} = a_{1,i-1}$, $z_{2i} = a_{0,i-1}$, $r_{2i-1} = y_{1,i-1}$, and $r_{2i} = y_{0,i-1}$ with the additional condition that $r_0 = z_0 = 0$. Then the Lemma II.1.1 equations can be encoded as

$$z_0 = 0, \quad z_1 = r_1, \quad z_k = r_k + 2z_{k-1} - z_{k-2}.$$

Let $g(x) = \sum_{k=0}^{\infty} z_k x^k$ be the generating function for $(z_k)_{k=0}^{\infty}$. Then

$$\begin{aligned} g(x) &= r_1 x + \sum_{k=2}^{\infty} z_k x^k \\ &= r_1 x + \sum_{k=2}^{\infty} (r_k + 2z_{k-1} - z_{k-2}) x^k \\ &= r_1 x + \sum_{k=2}^{\infty} r_k + 2x \sum_{k=2}^{\infty} z_{k-1} x^{k-1} - x^2 \sum_{k=2}^{\infty} z_{k-2} x^{k-2}. \end{aligned}$$

Note that the summations $\sum_{k=2}^{\infty} z_{k-1} x^{k-1}$ and $\sum_{k=2}^{\infty} z_{k-2} x^{k-2}$ can both be reindexed to show equivalence to $g(x)$. Thus,

$$g(x) = r_1 x + \sum_{k=2}^{\infty} r_k + 2x g(x) - x^2 g(x).$$

Solving the above equation for $g(x)$ yields

$$g(x) = \frac{\sum_{k=0}^{\infty} r_k}{x^2 - 2x + 1} = \frac{\sum_{k=0}^{\infty} r_k}{(x-1)^2} = \frac{1}{(1-x)^2} \sum_{k=0}^{\infty} r_k.$$

Note that the power series of $\frac{1}{(1-x)^2}$ is $\sum_{i=0}^{\infty}(i+1)x^i$, so

$$g(x) = \left(\sum_{i=0}^{\infty}(i+1)x^i\right)\left(\sum_{k=0}^{\infty} r_k\right).$$

This can be expanded to a single power series to yield

$$g(x) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}(i+1)r_{k-i}\right)x^k.$$

Thus,

$$z_k = \sum_{i=0}^{k}(i+1)r_{k-i}.$$

This can be broken up into two different cases to show the desired result. First, note that

$$a_{0,k} = z_{2k+2} = \sum_{i=0}^{2k+2}(i+1)r_{2k+2-i}.$$

The sum can be broken up into even and odd parts, so

$$\begin{aligned}
z_{2k+1} &= \sum_{i=0}^{k+1}(2i+1)r_{2k-2i+2} + \sum_{i=0}^{k}(2i+2)r_{2k-2i+1} \\
&= (2k+3)r_0 + \sum_{i=0}^{k}((2i+1)r_{2k-2i+2} + (2i+2)r_{2k-2i+1}) \\
&= \sum_{i=0}^{k}((2i+1)y_{0,k-i} + (2i+2)y_{1,k-i}).
\end{aligned}$$

Now, to prove the final part of the theorem, consider

$$\begin{aligned}
a_{1,k} = z_{2k+1} &= \sum_{i=0}^{2k+1}(i+1)r_{2k+1-i} \\
&= \sum_{i=0}^{k}(2i+1)r_{2k-2i+1} + \sum_{i=0}^{k}(2i+2)r_{2k-2i} \\
&= (2k+2)r_0 + (2k+1)r_1 + \sum_{i=0}^{k-1}(2i+1)r_{2k-2i+1} + \sum_{i=0}^{k}(2i+2)r_{2k-2i} \\
&= (2k+1)y_{1,0} + \sum_{i=0}^{k-1}((2i+1)y_{1,k-i} + (2i+2)y_{0,k-i-1}).
\end{aligned}$$

□

Note that Lemma II.1.2 implies that the $a_{i,k}$ are uniquely determined for a given M. Therefore, this gives an explicit formula for the coefficient of δ in $\mathrm{wt}(M)$ by considering $\sum_{k\geq 0} a_{0,k}$. This gives the following result.

Theorem II.1.3. *If* $M = \prod_{i\in I}\prod_{k\in\mathbb{Z}_{\geq 0}} Y_{i,k}^{y_{i,k}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$*, then coefficient of* δ *in* $\mathrm{wt}(M)$ *is*

$$\sum_{k\geq 0}\sum_{j=0}^{k}((2j+1)y_{0,k-j}+(2j+2)y_{1,k-j}).$$

Example II.1.4. Let $M = \tilde{f}_0\tilde{f}_1\tilde{f}_0\tilde{f}_0\tilde{f}_0\mathbf{1}$. Note that

$$M = A_{0,0}^{-3}A_{0,1}^{-1}A_{1,1}^{-1}\mathbf{1} = Y_{0,0}^{-3}Y_{0,1}^{-2}Y_{0,2}^{-1}Y_{1,1}^{5}Y_{1,2}\mathbf{1}.$$

Using the values of the $y_{i,k}$ shows that

$$\begin{aligned}
&\sum_{k\geq 0}\sum_{j=0}^{k}\big((2j+1)y_{0,k-j}+(2j+2)y_{1,k-j}\big)\\
&\qquad = (y_{0,0}+2y_{1,0})+(y_{0,1}+2y_{1,1}+3y_{0,0}+4y_{1,0})\\
&\qquad\quad +(y_{0,2}+2y_{1,2}+3y_{0,1}+4y_{1,1}+5y_{0,0}+6y_{1,0})\\
&\qquad\quad +(y_{0,3}+2y_{1,3}+3y_{0,2}+4y_{1,2}+5y_{0,1}+6y_{1,1}+7y_{0,0}+8y_{1,0})\\
&\qquad\quad +(y_{0,4}+2y_{1,4}+3y_{0,3}+4y_{1,3}+5y_{0,2}+6y_{1,2}+7y_{0,1}+8y_{1,1}+9y_{0,0}+10y_{1,0})+\cdots\\
&\qquad = (-3+2(0))+(-2+2(5)+3(-3)+4(0))\\
&\qquad\quad +(-1+2(1)+3(-2)+4(5)+5(-3)+6(0))\\
&\qquad\quad +(0+2(0)+3(-1)+4(1)+5(-2)+6(5)+7(-3)+8(0))\\
&\qquad\quad +(0+2(0)+3(0)+4(0)+5(-1)+6(1)+7(-2)+8(5)+9(-3)+10(0))+\cdots\\
&\qquad = -3-1+0+0+0+\cdots.
\end{aligned}$$

Thus, Theorem II.1.3 suggests the coefficient of δ in $\mathrm{wt}(M)$ is -4. This matches the number of $\tilde{f}_0$ applied to reach M, which is the expected result.

Example II.1.5. Let $M = \tilde{f}_0\tilde{f}_1\tilde{f}_1\tilde{f}_0\mathbf{1} = A_{0,0}^{-1}A_{0,1}^{-1}A_{1,0}^{-2}\mathbf{1} = Y_{0,0}^{-1}Y_{0,1}^{2}Y_{0,2}^{-1}\mathbf{1}$. Note that $\mathrm{wt}(M) = -2\delta$. Applying Lemma II.1.2 gives

$$a_{0,0} = y_{0,0} + 2y_{1,0} = -1,$$

$$a_{0,1} = y_{0,1} + 2y_{1,1} + 3y_{0,0} + 4y_{1,0} = 2 - 3 = -1,$$

$$a_{0,2} = y_{0,2} + 2y_{1,2} + 3y_{0,1} + 4y_{1,1} + 5y_{0,0} + 6y_{1,0} = -1 + 6 - 5 = 0.$$

Adding up these values shows that the coefficient of δ in $\mathrm{wt}(M)$ is -2 according to Theorem II.1.3, as expected.

II.2. The Result for Type $A_n^{(1)}$

This section will focus on the $U_q(A_n^{(1)})$-crystal $\widehat{\mathscr{M}}(\mathbf{1})$ for $n \geq 2$. Here, the choice of $(o_{ij})_{i\neq j}$ is given by $o_{0n} = 0$, $o_{n0} = 1$, and for all other values,

$$o_{ij} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

Thus, for $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$, the $A_{i,k}$ are given by

$$A_{i,k} = Y_{i,k}Y_{i,k+1}Y_{i-1,k+1}^{-1}Y_{i+1,k}^{-1}.$$

Unfortunately, in this more general case, the same method used for $U_q(A_1^{(1)})$ does not yield an explicit formula for the weight function. Consider the following result.

Lemma II.2.1. *If* $M = \prod_{k\in\mathbb{Z}_{\geq 0}}\prod_{i\in I} Y_{i,k}^{y_{i,k}}\mathbf{1} = \prod_{k\in\mathbb{Z}_{\geq 0}}\prod_{i\in I} A_{i,k}^{a_{i,k}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$, *we have*

$$a_{i,k} - a_{i-1,k} = \sum_{n=0}^{k} y_{i+n,k-n}$$

for all $i \in I$ *and* $k \in \mathbb{Z}_{\geq 0}$.

Proof. This will be a proof by induction on k. As a base case, note that when $k = 0$,

$$a_{i,0} - a_{i-1,0} = y_{i,0}.$$

To show that this holds, note that the only terms in $\prod_{k\in\mathbb{Z}_{\geq 0}} \prod_{i\in I} A_{i,k}^{a_{i,k}} \mathbf{1}$ that contain any $Y_{i,0}$ are $A_{i,0}$ and $A_{i-1,0}$. These contribute $Y_{i,0}$ and $Y_{i,0}^{-1}$ respectively to the overall product, so certainly $y_{i,0} = a_{i,0} - a_{i-1,0}$.

Next, for the sake of induction, assume that the result holds for all values of i and a given value k. Then note that

$$\sum_{n=0}^{k+1} y_{i+n,k+1-n} = y_{i,k+1} + \sum_{n=1}^{k+1} y_{i+n,k+1-n} = y_{i,k+1} + \sum_{n=0}^{k} y_{i+n+1,k-n}.$$

By the inductive assumption, this is equal to

$$y_{i,k+1} + a_{i+1,k} - a_{i,k}.$$

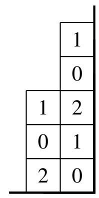
Now consider $y_{i,k+1}$. Note that $y_{i,k+1} = a_{i,k} - a_{i+1,k} + a_{i,k+1} - a_{i-1,k+1}$. This equation can be obtained by considering the values of $\prod_{m\in\mathbb{Z}_{\geq 0}} \prod_{j\in I} A_{j,m}^{a_{j,m}}$ that contribute to $Y_{i,k}$. Using this equation yields

$$\begin{aligned}\sum_{m=0}^{k+1} y_{i+m,k+1-m} &= a_{i,k} - a_{i+1,k} + a_{i,k+1} - a_{i-1,k+1} + a_{i+1,k} - a_{i,k} \\ &= a_{i,k+1} - a_{i-1,k+1},\end{aligned}$$

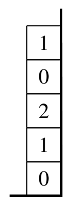
so the result holds by induction. □

Consider the general solution to the above equations for a fixed k. Note that if a specific solution is given by integers r_i such that $a_{i,k} = r_i$ for $i \in I$, then for any $t \in \mathbb{Z}$, another solution is given by $a_{i,k} = r_i + t$ for all $i \in I$. Consider now the following motivating example.

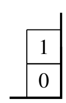
Example II.2.2. In type $A_2^{(1)}$, let $M = \tilde{f}_1\tilde{f}_0\mathbf{1} = Y_{0,0}^{-1}Y_{1,1}^{-1}Y_{2,0}Y_{2,1}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$. This can be written as $A_{0,0}^{-2}A_{1,0}^{-2}A_{2,0}^{-1}A_{0,1}^{-1}A_{1,1}^{-1}A_{2,1}^{-1}\mathbf{1}$. Consider the following generalized Young wall

$Y =$  $\in \mathscr{F}(\infty)$.

This is the Young wall that, if it were in $\mathscr{Y}(\infty)$, would be sent to M by the isomorphism between $\widehat{\mathscr{M}}(\mathbf{1})$ and $\mathscr{Y}(\infty)$. Note that the second column from the right has a removable δ. Removing it yields the Young wall

 .

This new Young wall also has a removable δ, this time in the first column. Removing it yields

 .

This is now an element of $\mathscr{Y}(\infty)$. Considered as an element of $\widehat{\mathscr{M}}(\mathbf{1})$, this is $A_{0,0}^{-1}A_{1,0}^{-1}\mathbf{1}$. Expanding this gives $Y_{0,0}^{-1}Y_{1,1}^{-1}Y_{2,0}Y_{2,1}\mathbf{1}$, so this Young wall actually corresponds to the original M.

To consider why this happened, note that eliminating a removable δ from the kth column of a Young wall is the same as shifting the $a_{i,k}$ down by one. It can be easily checked that, in type $A_n^{(1)}$, $\prod_{i\in I} A_{i,k}\mathbf{1} = \mathbf{1}$ for any $k \in \mathbb{Z}_{\geq 0}$. Thus, eliminating a removable δ does not change which monomial a generalized Young wall corresponds to. Thus, the weight function on $\mathscr{Y}(\infty)$ can here be used to calculate the coefficient of δ on the weight function of elements of $\widehat{\mathscr{M}}(\mathbf{1})$.

This idea gives rise to the following algorithm for computing the δ coefficient.

Theorem II.2.3. *Let* $M = \prod_{k\in\mathbb{Z}_{\geq 0}} \prod_{i\in I} Y_{i,k}^{y_{i,k}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$. *Then the coefficient of* δ *in* $\mathrm{wt}(M)$ *can be found using the following algorithm.*

1. *Find the maximum value* m *such that there exists an* $i \in I$ *with* $y_{i,m+1} \neq 0$.

2. Find the unique solution to $\{a_{i,m}\}_{i\in I}$ such that $a_{i,m} \in \mathbb{Z}_{\leq 0}$ for all i and there exists an i such that $a_{i,m} = 0$. Let these be the values for $a_{i,m}$. Define a new variable k such that $k = m-1$.

3. Find the maximal solution to $\{a_{i,k}\}_{i\in I}$ such that $a_{i,k} \in \mathbb{Z}_{\leq 0}$ and $a_{i,k} \leq a_{i-1,k+1}$ for all i. Decrease k by 1.

4. Repeat step 3 until $k = -1$.

After finding the values of $a_{i,k}$ in this manner, the coefficient of δ in $\mathrm{wt}(M)$ is given by $\sum_{k=0}^{m} a_{0,k}$.

Proof. First note that in step 2, such an integral solution to the equations from Lemma II.2.1 exists since $M \in \widehat{\mathscr{M}(\mathbf{1})}$ so must be expressible via the $A_{i,k}$. Furthermore, given a solution $(c_i)_{i\in I}$ of integers to the system of equations, any other solution $(c'_i)_{i\in I}$ of integers is given by some integral shift of the c_i. This can be seen by first noting that there exists an integer t such that $c'_0 = c_0 + t$. Then note that

$$
\begin{aligned}
c_0 - c_n &= c'_0 - c'_n \\
c_0 - c_n &= (c_0 + t) - c'_n \\
-c_n &= t - c'_n \\
c'_n &= c_n + t.
\end{aligned}
$$

This can be repeated to show that $c'_i = c_i + t$ for all $i \in I$. Thus, the solution to step 2 is the integral shift of the first solution such that some value is 0 and the rest are nonpositive.

Showing the existence and uniqueness of the solution in step 3 is nearly identical to the argument used for step 2. It now suffices to show that the solution obtained by this algorithm represents the correct expression for M corresponding to a proper reduced generalized Young wall.

Given any nonpositive solution to the $a_{i,m}$, note that if none of the values are 0, the corresponding Young wall has a removable δ. Thus, the solution here should certainly be the one given by the algorithm.

For any $a_{i,k-1}$, note that if any $a_{i,k-1} > a_{i-1,k}$, then the corresponding Young wall will not be proper. Therefore, it is certainly true that $a_{i,k-1} \leq a_{i-1,k}$. Furthermore, if $a_{i,k-1} < a_{i-1,k}$ for all i, then the corresponding Young wall has a removable δ in its $(k-1)$th row and so is not reduced. Thus, it must be the maximal solution, where for at least one i, $a_{i,k-1} = a_{i-1,k}$.

Since the $a_{0,k}$ values correspond to the 0-boxes in the Young wall, the coefficient of δ in $\mathrm{wt}(M)$ is certainly $\sum_{k=0}^{m} a_{0,k}$. □

II.2.1. Examples

Example II.2.4. Note that in $U_q(A_4^{(1)})$,

$$\tilde{f}_0\tilde{f}_1\tilde{f}_3\tilde{f}_0\tilde{f}_4\tilde{f}_0\mathbf{1} = A_{0,0}^{-3}A_{1,0}^{-1}A_{3,2}^{-1}A_{4,1}^{-1}\mathbf{1} = Y_{0,0}^{-3}Y_{0,1}^{-1}Y_{1,0}^{2}Y_{1,1}^{-1}Y_{2,0}Y_{2,3}Y_{3,3}^{-1}Y_{4,1}^{2}\mathbf{1}.$$

Applying Theorem II.2.3, note that the desired value m is 2. Step (2) of the algorithm says that the correct values of $a_{i,2}$ $(i \in I)$ are solutions to the system

$$\begin{aligned}
a_{0,2} - a_{4,2} &= y_{0,2} + y_{1,1} + y_{2,0} = 0,\\
a_{1,2} - a_{0,2} &= y_{1,2} + y_{2,1} + y_{3,0} = 0,\\
a_{2,2} - a_{1,2} &= y_{2,2} + y_{3,1} + y_{4,0} = 0,\\
a_{3,2} - a_{2,2} &= y_{3,2} + y_{4,1} + y_{0,0} = -1,\\
a_{4,2} - a_{3,2} &= y_{4,2} + y_{0,1} + y_{1,0} = 1.
\end{aligned}$$

The general solution to this system is (for any $t \in \mathbb{Z}$) $a_{0,2} = t$, $a_{1,2} = t$, $a_{2,2} = t$, $a_{3,2} = t-1$, and $a_{4,2} = t$. Note that the maximum solution to this system with nonpositive values is $a_{3,2} = -1$ and $a_{i,2} = 0$ for all other i. This matches what we know the solution to be.

Similarly, for $k = 1$, the values of $a_{i,1}$ $(i \in I)$ are a solution to the system

$$\begin{aligned}
a_{0,1} - a_{4,1} &= y_{0,1} + y_{1,0} = 1,\\
a_{1,1} - a_{0,1} &= y_{1,1} + y_{2,0} = 0,
\end{aligned}$$

$$a_{2,1} - a_{1,1} = y_{2,1} + y_{3,0} = 0,$$

$$a_{3,1} - a_{2,1} = y_{3,1} + y_{4,0} = 0,$$

$$a_{4,1} - a_{3,1} = y_{4,1} + y_{0,0} = -1.$$

The maximal solution to this system such that $a_{4,1} \leq -1$ and $a_{i,1} \leq 0$ for all other i is $a_{4,1} = -1$ and $a_{i,1} = 0$ for all other i. This matches what we know the solution to be.

Finally, for $k = 0$, the values for $a_{i,0}$ $(i \in I)$ are solutions to the system

$$a_{0,0} - a_{4,0} = y_{0,0} = -3,$$

$$a_{1,0} - a_{0,0} = y_{1,0} = 2,$$

$$a_{2,0} - a_{1,0} = y_{2,0} = 1,$$

$$a_{3,0} - a_{2,0} = y_{3,0} = 0,$$

$$a_{4,0} - a_{3,0} = y_{4,0} = 0.$$

Note that the maximal solution to this such that $a_{0,0} \leq -1$ and $a_{i,0} \leq 0$ for all other i is $a_{0,0} = -3$, $a_{1,0} = -1$, $a_{2,0} = 0$, $a_{3,0} = 0$, and $a_{4,0} = 0$. Therefore, all of the values according to Theorem II.2.3 are as expected.

CHAPTER III

TYPE B WEIGHT FUNCTION

III.1. The Result for $B_3^{(1)}$

This section will focus on the $U_q(B_3^{(1)})$-crystal $\widehat{\mathscr{M}}(\mathbf{1})$. Here, define $(o_{ij})_{i\neq j}$ by

$$o_{ij} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

The first three levels of the crystal graph are given in terms of both the Y-variables (Figure 6) and the A-variables (Figure 7).

Lemma III.1.1. *For* $M = \prod_{i\in I}\prod_{k\geq 0} Y_{i,k}^{y_{i,k}}\mathbf{1} = \prod_{i\in I}\prod_{k\geq 0} A_{i,k}^{a_{i,k}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$*, each of the following hold for* $k \geq 0$ *(with the convention that* $a_{i,-1} = 0$ *for all* $i \in I$*):*

$$a_{0,k} = y_{0,k} + a_{2,k-1} - a_{0,k-1},$$

$$a_{1,k} = y_{1,k} + a_{2,k-1} - a_{1,k-1},$$

$$a_{2,k} = y_{2,k} + a_{0,k} + a_{1,k} + a_{3,k-1} - a_{2,k-1},$$

$$a_{3,k} = y_{3,k} + 2a_{2,k} - a_{3,k-1}.$$

Proof. Consider which $A_{i,k}$ contribute $Y_{0,\ell}$ to M. They are exactly $A_{0,\ell}$, $A_{0,\ell-1}$, and $A_{2,\ell-1}$. Expanding these and equating powers with those of $Y_{0,\ell}$ gives

$$y_{0,\ell} = a_{0,\ell} + a_{0,\ell-1} - a_{2,\ell-1},$$

which gives the first desired equality. The other three follow similarly. □

Lemma III.1.2. *For any* $m \in \mathbb{Z}_{\geq 0}$*,*

$$\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m+1}{2} \right\rfloor = m.$$

Proof. Note that $\left\lfloor \frac{m}{2} \right\rfloor$ is the number of positive even integers less than or equal to m. Similarly, $\left\lfloor \frac{m+1}{2} \right\rfloor$ is the number of positive odd integers less than or equal to m. Summing these values gives the number of integers less than or equal to m, which is exactly m. □

Lemma III.1.3. *Given* $M = \prod_{i\in I}\prod_{m\geq 0} Y_{i,m}^{y_{i,m}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$, *the solution to* $M = \prod_{i\in I}\prod_{m\geq 0} A_{i,m}^{a_{i,m}}\mathbf{1}$ *is*

$$\begin{aligned}
a_{0,m} &= \sum_{k=0}^{m} \left(2\left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k-1}{2} \right\rfloor\right) y_{0,m-k} + \left\lfloor \frac{k+1}{2} \right\rfloor y_{1,m-k} + k y_{2,m-k} + \left\lfloor \frac{k}{2} \right\rfloor y_{3,m-k},\\
a_{1,m} &= \sum_{k=0}^{m} \left\lfloor \frac{k+1}{2} \right\rfloor y_{0,m-k} + \left(2\left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k-1}{2} \right\rfloor\right) y_{1,m-k} + k y_{2,m-k} + \left\lfloor \frac{k}{2} \right\rfloor y_{3,m-k},\\
a_{2,m} &= \sum_{k=0}^{m} (k+1) y_{0,m-k} + (k+1) y_{1,m-k} + (2k+1) y_{2,m-k} + k y_{3,m-k},\\
a_{3,m} &= \sum_{k=0}^{m} 2\left\lfloor \frac{k+2}{2} \right\rfloor y_{0,m-k} + 2\left\lfloor \frac{k+2}{2} \right\rfloor y_{1,m-k} + (2k+2) y_{2,m-k} + \left(2\left\lfloor \frac{k}{2} \right\rfloor + 1\right) y_{3,m-k}.
\end{aligned}$$

Proof. This will be a proof by induction on m. First, note that the base case of $a_{0,0} = y_{0,0}$, $a_{1,0} = y_{1,0}$, $a_{2,0} = y_{0,0} + y_{1,0} + y_{2,0}$ and $a_{3,0} = y_{3,0} + 2y_{0,0} + 2y_{1,0} + 2y_{2,0}$ each hold by the Lemma III.1.1.

Now assume for the sake of induction that for a fixed $m \in \mathbb{Z}_{\geq 0}$ the result holds for $a_{i,m}$ for all $i \in \{0,1,2,3\}$. Then the relation from Lemma III.1.1 gives

$$\begin{aligned}
a_{0,m+1} = y_{0,m+1} + \sum_{k=0}^{m} (k+1) y_{0,m-k} + (k+1) y_{1,m-k} + (2k+1) y_{2,m-k} + k y_{3,m-k}\\
- \sum_{k=0}^{m} \left(2\left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k-1}{2} \right\rfloor\right) y_{0,m-k} + \left\lfloor \frac{k+1}{2} \right\rfloor y_{1,m-k} + k y_{2,m-k} + \left\lfloor \frac{k}{2} \right\rfloor y_{3,m-k}.
\end{aligned}$$

Since the desired relation is (by evaluating the $k = 0$ term and reindexing the sum)

$$\begin{aligned}
a_{0,m+1} =& y_{0,m+1} + \sum_{k=0}^{m} \left(2\left\lfloor \frac{k+1}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor\right) y_{0,m-k}\\
&+ \left\lfloor \frac{k+2}{2} \right\rfloor y_{1,m-k} + (k+1) y_{2,m-k} + \left\lfloor \frac{k+1}{2} \right\rfloor y_{2,m-k},
\end{aligned}$$

it suffices to prove the following four relations:

$$2\left\lfloor\frac{k+1}{2}\right\rfloor-\left\lfloor\frac{k}{2}\right\rfloor=k+1-2\left\lfloor\frac{k}{2}\right\rfloor+\left\lfloor\frac{k-1}{2}\right\rfloor,$$
$$\left\lfloor\frac{k+2}{2}\right\rfloor=k+1-\left\lfloor\frac{k+1}{2}\right\rfloor,$$
$$k+1=2k+1-k,$$
$$\left\lfloor\frac{k+1}{2}\right\rfloor=k-\left\lfloor\frac{k}{2}\right\rfloor.$$

The third relation holds trivially and the second and fourth relations are each immediate results from Lemma III.1.2. To prove the first relation, note that

$$\begin{aligned}2\left\lfloor\frac{k+1}{2}\right\rfloor-\left\lfloor\frac{k}{2}\right\rfloor+2\left\lfloor\frac{k}{2}\right\rfloor-\left\lfloor\frac{k-1}{2}\right\rfloor&=2\left(\left\lfloor\frac{k+1}{2}\right\rfloor+\left\lfloor\frac{k}{2}\right\rfloor\right)-\left(\left\lfloor\frac{k}{2}\right\rfloor+\left\lfloor\frac{k-1}{2}\right\rfloor\right)\\&=2k-(k-1)\\&=k+1,\end{aligned}$$

which is equivalent to the first statement. Thus, the result holds for $a_{0,m+1}$.

Now consider similarly $a_{1,m+1}$. Using a method identical to what was used for $a_{0,m+1}$, it can be seen that this case holds if the following four relations hold:

$$\left\lfloor\frac{k+2}{2}\right\rfloor=k+1-\left\lfloor\frac{k+1}{2}\right\rfloor,$$
$$2\left\lfloor\frac{k+1}{2}\right\rfloor-\left\lfloor\frac{k}{2}\right\rfloor=k+1-2\left\lfloor\frac{k}{2}\right\rfloor+\left\lfloor\frac{k-1}{2}\right\rfloor,$$
$$k+1=2k+1-k,$$
$$\left\lfloor\frac{k+1}{2}\right\rfloor=k-\left\lfloor\frac{k}{2}\right\rfloor.$$

Since each of these are identical to a relation used to prove the result for $a_{0,m+1}$, the result holds for $a_{1,m+1}$ as well.

Next, to show that the theorem holds for $a_{2,m+1}$, note first that by Lemma III.1.1:

$$\begin{aligned}
&a_{2,m+1} = y_{2,m+1} \\
&+ \sum_{k=0}^{m+1} \left(2\left\lfloor\frac{k}{2}\right\rfloor - \left\lfloor\frac{k-1}{2}\right\rfloor\right) y_{0,m+1-k} + \left\lfloor\frac{k+1}{2}\right\rfloor y_{1,m+1-k} + k y_{2,m+1-k} + \left\lfloor\frac{k}{2}\right\rfloor y_{3,m+1-k} \\
&+ \sum_{k=0}^{m+1} \left\lfloor\frac{k+1}{2}\right\rfloor y_{0,m+1-k} + \left(2\left\lfloor\frac{k}{2}\right\rfloor - \left\lfloor\frac{k-1}{2}\right\rfloor\right) y_{1,m+1-k} + k y_{2,m+1-k} + \left\lfloor\frac{k}{2}\right\rfloor y_{3,m+1-k} \\
&+ \sum_{k=0}^{m} 2\left\lfloor\frac{k+2}{2}\right\rfloor y_{0,m-k} + 2\left\lfloor\frac{k+2}{2}\right\rfloor y_{1,m-k} + (2k+2) y_{2,m-k} + (2\left\lfloor\frac{k}{2}\right\rfloor + 1) y_{3,m-k} \\
&- \sum_{k=0}^{m} (k+1) y_{0,m-k} + (k+1) y_{1,m-k} + (2k+1) y_{2,m-k} + k y_{3,m-k}.
\end{aligned}$$

Pulling out the $k=0$ term of the first two sums and reindexing so that each of the sums match gives

$$\begin{aligned}
&a_{2,m+1} = y_{0,m+1} + y_{1,m+1} + y_{2,m+1} \\
&+ \sum_{k=0}^{m} \left(2\left\lfloor\frac{k+1}{2}\right\rfloor - \left\lfloor\frac{k}{2}\right\rfloor\right) y_{0,m-k} + \left\lfloor\frac{k+2}{2}\right\rfloor y_{1,m-k} + (k+1) y_{2,m-k} + \left\lfloor\frac{k+1}{2}\right\rfloor y_{2,m-k} \\
&+ \sum_{k=0}^{m} \left\lfloor\frac{k+2}{2}\right\rfloor y_{0,m-k} + \left(2\left\lfloor\frac{k+1}{2}\right\rfloor - \left\lfloor\frac{k}{2}\right\rfloor\right) y_{1,m-k} + (k+1) y_{2,m-k} + \left\lfloor\frac{k+1}{2}\right\rfloor y_{2,m-k} \\
&+ \sum_{k=0}^{m} 2\left\lfloor\frac{k+2}{2}\right\rfloor y_{0,m-k} + 2\left\lfloor\frac{k+2}{2}\right\rfloor y_{1,m-k} + (2k+2) y_{2,m-k} + \left(2\left\lfloor\frac{k}{2}\right\rfloor + 1\right) y_{3,m-k} \\
&- \sum_{k=0}^{m} (k+1) y_{0,m-k} + (k+1) y_{1,m-k} + (2k+1) y_{2,m-k} + k y_{3,m-k}.
\end{aligned}$$

Since the desired result is equivalent to (by evaluating the $k=0$ term and reindexing the sum)

$$\begin{aligned}
a_{2,m+1} =& y_{0,m+1} + y_{1,m+1} + y_{2,m+1} \\
&+ \sum_{k=0}^{m} (k+2) y_{0,m-k} + (k+2) y_{1,m-k} + (2k+3) y_{2,m-k} + (k+1) y_{3,m-k},
\end{aligned}$$

the four identities needed to show that the result holds for $a_{2,m+1}$ are

$$k+2 = 2\left\lfloor\frac{k+1}{2}\right\rfloor - \left\lfloor\frac{k}{2}\right\rfloor + \left\lfloor\frac{k+2}{2}\right\rfloor + 2\left\lfloor\frac{k+2}{2}\right\rfloor - k - 1,$$

$$k+2 = \left\lfloor \frac{k+2}{2} \right\rfloor + 2\left\lfloor \frac{k+2}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor + 2\left\lfloor \frac{k+2}{2} \right\rfloor - k - 1,$$
$$2k+3 = k+1+k+1+2k+2-2k-1,$$
$$k+1 = \left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor + 2\left\lfloor \frac{k}{2} \right\rfloor + 1 - k.$$

The third statement is trivial and the fourth statement is a direct application of Lemma III.1.2. To show the first and second equalities (which are both the same statement written different ways), note that

$$\begin{aligned}
&2\left\lfloor \frac{k+1}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor + 2\left\lfloor \frac{k+2}{2} \right\rfloor - k - 1 \\
&= 2\left(\left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor\right) - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor - k - 1 \\
&= 2k+2 - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor - k - 1 \\
&= k+1 - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor.
\end{aligned}$$

Thus, proving the statement reduces to showing that $-\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor = 1$. By Lemma III.1.2, $-\left\lfloor \frac{k}{2} \right\rfloor = -k + \left\lfloor \frac{k+1}{2} \right\rfloor$, so the statement further reduces to showing that $-k + \left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor = 1$, which can be shown by another application of Lemma III.1.2. Therefore, each of the four desired identities holds, so the result holds for $a_{2,m+1}$.

Finally, to show the result for $a_{3,m+1}$, an identical process to the one used for $a_{2,m+1}$ can be used to see that the four desired identities are

$$2\left\lfloor \frac{k+3}{2} \right\rfloor = 2(k+2) - 2\left\lfloor \frac{k+2}{2} \right\rfloor,$$
$$2\left\lfloor \frac{k+3}{2} \right\rfloor = 2(k+2) - 2\left\lfloor \frac{k+2}{2} \right\rfloor,$$
$$2k+4 = 2(2k+3) - 2k + 2,$$
$$2\left\lfloor \frac{k+1}{2} \right\rfloor + 1 = 2(k+1) - 2\left\lfloor \frac{k}{2} \right\rfloor - 1.$$

However, the third statement is trivial and each of the remaining statements are a straightforward applications of Lemma III.1.2, so each of these statements hold, and therefore so does the desired result for $a_{3,m+1}$.

Thus, by induction, the theorem holds. $\square$

Note that Lemma III.1.3 shows that the $a_{i,k}$ are uniquely determined, the $\sum_{k\geq 0} a_{0,k}$ gives the δ coefficient of the weight function here. This gives the following proposition.

Theorem III.1.4. *Let* $M = \prod_{i\in I}\prod_{k\in\mathbb{Z}_{\geq 0}} Y_{i,k}^{y_{i,k}}\mathbf{1} \in \widehat{\mathscr{M}}(\mathbf{1})$. *Then the coefficient of* δ *in* $\mathrm{wt}(M)$ *is*

$$\sum_{m\geq 0}\left(\sum_{k=0}^{m}\left(2\left\lfloor\frac{k}{2}\right\rfloor - \left\lfloor\frac{k-1}{2}\right\rfloor\right)y_{0,m-k} + \left\lfloor\frac{k+1}{2}\right\rfloor y_{1,m-k} + ky_{2,m-k} + \left\lfloor\frac{k}{2}\right\rfloor y_{3,m-k}\right).$$

Example III.1.5. Note that $\tilde{f}_0\tilde{f}_1\tilde{f}_2\tilde{f}_3\mathbf{1} = Y_{0,3}^{-1}Y_{1,3}^{-1}Y_{2,2}Y_{3,0}^{-1}Y_{3,1}\mathbf{1}$ and

$$\mathrm{wt}(\tilde{f}_0\tilde{f}_1\tilde{f}_2\tilde{f}_3\mathbf{1}) = -\Lambda_0 - \Lambda_1 + \Lambda_2 - \delta.$$

Note that the coefficient of δ in $\mathrm{wt}(M)$ is -1. Indeed, consider the values of $a_{0,m}$ for each m. For $m \geq 3$, $a_{0,m} = 0$ since the largest nonzero $y_{i,k}$ is $y_{1,3}$. Now, applying Lemma III.1.3 (and ignoring the values for which $y_{i,k} = 0$) shows that

$$\begin{aligned}
a_{0,0} &= \left\lfloor\frac{0}{2}\right\rfloor y_{3,0} = 0,\\
a_{1,0} &= \left\lfloor\frac{0}{2}\right\rfloor y_{3,1} + \left\lfloor\frac{1}{2}\right\rfloor y_{3,0} = 0,\\
a_{2,0} &= 0y_{2,2} + \left\lfloor\frac{1}{2}\right\rfloor y_{3,1} + \left\lfloor\frac{2}{2}\right\rfloor y_{3,0} = -1.
\end{aligned}$$

Thus, by adding these up, Theorem III.1.4 gives the coefficient of δ in $\mathrm{wt}(\tilde{f}_0\tilde{f}_1\tilde{f}_2\tilde{f}_3\mathbf{1})$ to be -1, as expected.

III.2. An Attempt at $B_4^{(1)}$

This section will focus on the $U_q(B_4^{(1)})$-crystal $\widehat{\mathscr{M}}(\mathbf{1})$. Here, define $(o_{ij})_{i\neq j}$ by

$$o_{ij} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

In an attempt to find a similar result for the $U_q(B_4^{(1)})$-crystal $\widehat{\mathscr{M}}(\mathbf{1})$, note that the analogous defining relations to Lemma III.1.1 are

$$\begin{aligned}
a_{0,m} &= y_{0,m} + a_{2,m-1} - a_{0,m-1}, \\
a_{1,m} &= y_{1,m} + a_{2,m-1} - a_{1,m-1}, \\
a_{2,m} &= y_{2,m} + a_{0,m} + a_{1,m} + a_{3,m-1} - a_{2,m-1}, \\
a_{3,m} &= y_{3,m} + a_{2,m} + a_{4,m-1} - a_{3,m-1}, \\
a_{4,m} &= y_{4,m} + 2a_{3,m} - a_{4,m-1}.
\end{aligned}$$

Note that, if

$$a_{0,m} = \sum_{k=0}^{m} a_k y_{0,m-k} + b_k y_{1,m-k} + c_k y_{2,m-k} + d_k y_{3,m-k} + e_k y_{4,m-k},$$

(and given similar recurrence relations to each $a_{i,m}$), the first k terms of each sequence can be manually computed. This can be done by first noting that

$$\begin{aligned}
a_{0,0} &= y_{0,0}, & a_{3,0} &= y_{0,0} + y_{1,0} + y_{2,0} + y_{3,0}, \\
a_{1,0} &= y_{1,0}, & a_{4,0} &= 2y_{0,0} + 2y_{1,0} + 2y_{2,0} + 2y_{3,0} + y_{4,0}, \\
a_{2,0} &= y_{0,0} + y_{1,0} + y_{2,0}.
\end{aligned}$$

Then, the known first terms of each sequence can be plugged into the analogous Lemma III.1.1 relations to generate each coefficient. The following code can be used in SAGEMATH [7] to compute the first 21 values of the sequences $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$:

```
sage: def coefficientsB4(n):
....:     a = vector([1,0,0,0,0])
....:     b = vector([0,1,0,0,0])
....:     c = vector([1,1,1,0,0])
....:     d = vector([1,1,1,1,0])
....:     e = vector([2,2,2,2,1])
....:     print['k=',0, 'a_k=', a[0], 'b_k=',a[1]]
....:     for i in range(n):
....:         a = c-a
....:         b = c-b
....:         c = a+b-c+d
....:         d = c+e-d
....:         e = 2*d-e
....:         print ['k=',i+1, 'a_k=', a[0], 'b_k=',a[1]]
....:
sage: coefficientsB4(20)
```

$$(a_k)_{k=0}^{20} = (1,0,1,1,2,1,3,2,3,3,4,3,5,4,5,5,6,5,7,6,7),$$

$$(b_k)_{k=0}^{20} = (0,1,0,2,1,2,2,3,2,4,3,4,4,5,4,6,5,6,6,7,6).$$

Note that each of these sequences is not obviously as simple as the sequences needed for $B_3^{(1)}$. Therefore, while the same method of finding sequences that generate the coefficients may work here, it is not immediately apparent how they would do so. In particular, the Online Encyclopedia of Integer Sequences [6] note that these first terms of a_k are consistent with the power series expansion of $\frac{1+x^4}{(1-x^2)(1-x^3)}$. However, b_k does not have any sequence associated with it on the Online Encyclopedia of Integer Sequences, so it seems likely that a different method would be necessary to find an explicit equation for the δ-coefficient of the weight function for $B_4^{(1)}$.

```
sage: M = crystals.infinity.NakajimaMonomials(['B',3,1])
sage: S = M.subcrystal(max_depth=2)
sage: G = M.digraph(subset=S)
sage: latex(G)
```

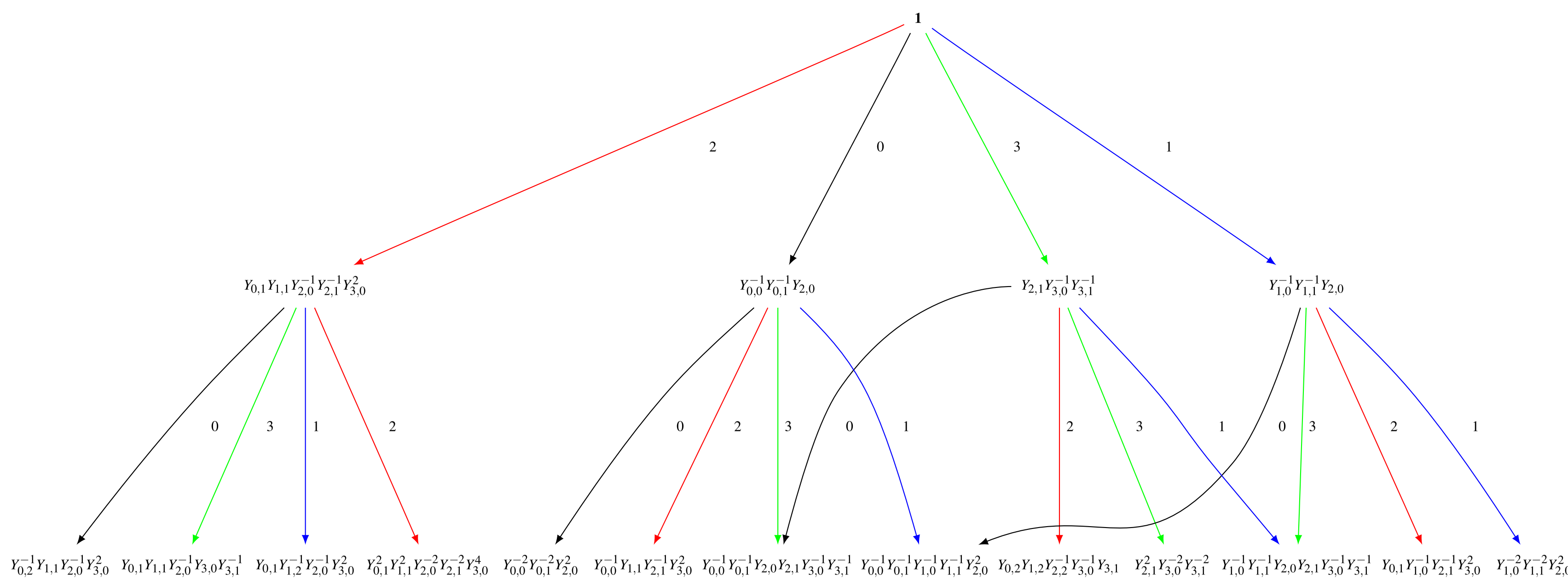


Figure 6. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $B_3^{(1)}$ in terms of the Y-variables, created using SAGEMATH [7].

```
sage: M = crystals.infinity.NakajimaMonomials(['B',3,1])
sage: M.set_variables("A")
sage: S = M.subcrystal(max_depth=2)
sage: G = M.digraph(subset=S)
sage: latex(G)
```

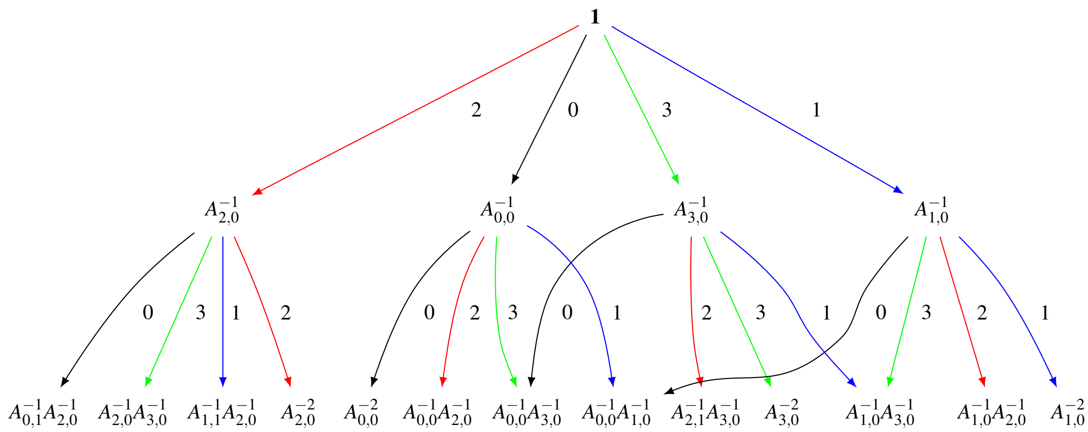


Figure 7. The top part of the crystal $\widehat{\mathscr{M}}(\mathbf{1})$ in type $B_3^{(1)}$ in terms of the A-variables, created using SAGEMATH [7].

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