THE ANNIHILATING-IDEAL GRAPH OF COMMUTATIVE SEMIGROUPS

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ABSTRACT

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by Anna Schneider

In this paper, we associate an undirected graph $\mathbb{AG}(S)$, the *annihilating-ideal graph*, to a commutative semigroup S. This graph has vertex set $\mathbb{A}^*(S) = \mathbb{A}(S) \setminus \{(0)\}$, where $\mathbb{A}(S)$ is the set of proper ideals of S with nonzero annihilator. Two distinct vertices $I, J \in \mathbb{A}^*(S)$ are defined to be adjacent in $\mathbb{AG}(S)$ if and only if $IJ = (0)$, the zero ideal. We begin by discussing conditions for ensuring a finite graph, as well as when each nonzero, proper ideal of S is an element of $\mathbb{A}^*(S)$. Connections are drawn between $\mathbb{AG}(S)$ and $\Gamma(S)$, the well-known zero divisor graph, and the connectivity, diameter, and girth of $\mathbb{AG}(S)$ are described. Then, we investigate the shape of $\mathbb{AG}(S)$ with results characterizing semigroups S for which $\mathbb{AG}(S)$ is a complete or star graph. Finally, we investigate the coloring of $\mathbb{AG}(S)$, showing that $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S))$ for each reduced semigroup and null semigroup, and giving upper and lower bounds for $\chi(\mathbb{AG}(S))$ for a general commutative semigroup.

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CHAPTER I

INTRODUCTION

In recent years, many different graphs have been assigned to rings and semigroups in order to study the interplay between the algebraic structure of the given object and the graph-theoretic properties of the graph to which it corresponds. This began with Beck in 1988 [6], who investigated assigning a graph to a commutative ring R by labeling vertices with the elements of R, in which two vertices x, y are adjacent in case $xy = 0$. Beck was mainly interested in the coloring of this graph, and this investigation of colorings of a commutative ring was continued by Anderson and Naseer in 1993 [4].

In 1999, Anderson and Livingston modified Beck's graph in [3], by restricting the vertex set to only the set of zero divisors, $Z(R)$, of a commutative ring R. Anderson and Livingston associated the graph $\Gamma(R)$ to the commutative ring R with vertex set $Z(R)$ and the same adjacency condition, $x, y \in Z(R)$ adjacent in case $xy = 0$. This is known as the *zero-divisor graph* of R, and its structure and colorings have been extensively studied.

This same definition of a zero-divisor graph was then applied to commutative semigroups with zero by F. DeMeyer, McKenzie, and K. Schneider in 2002, [10]. For a commutative semigroup S, the graph $\Gamma(S)$ has been extensively studied in similar methods to $\Gamma(R)$, with some surprisingly similar as well as some strikingly different results.

More recently, Behboodi and Rakeei have extended this idea by defining a graph of a commutative ring whose vertices are ideals of the ring, rather than elements, in [7] and [8]. Since the structure of a commutative ring R is more closely tied to the behavior of its ideals than its elements, this is a logical modification. For a commutative ring R with identity, let I be a proper ideal of R. Behboodi and Rakeei define I as an *annihilating-ideal* of R if there exists a nonzero ideal J of R such that $IJ = (0)$, the zero ideal. The set of all annihilating-ideals is denoted $\mathbb{A}(R)$, and the *annihilating-ideal graph* of R, as defined by Behboodi and Rakeei, is a graph with vertex set $\mathbb{A}^*(R) = \mathbb{A}(R) \setminus \{(0)\}$ where distinct vertices I and J are adjacent if and only if $IJ = (0)$. This annihilating-ideal graph is denoted $\mathbb{AG}(R)$.

In their work, Behboodi and Rakeei pursue similar results on annihilating-ideal graphs of commutative rings to those of zero-divisor graphs, seeking information on the finiteness, connectivity, classification, and coloring of $\mathbb{AG}(R)$, as well as such graph-theoretic properties' relations to the algebraic structure of R and its set of ideals. These results, particularly those related to coloring, are more deeply investigated by Aalipour, Akbari, Nikandish, Nikmehr, and Shaveisi [1].

Just as the definition of a zero-divisor graph of a commutative ring was applied to a commutative semigroup, in this paper we apply the given defintion of the annihilating-ideal of a commutative ring to a commutative semigroup, and investigate the interplay between the algebraic structure of a semigroup and the properties of such a graph defined on its ideals. The relationship between $\Gamma(S)$ and $\mathbb{AG}(S)$ is discussed, as are semigroups S for which $\mathbb{AG}(S)$ is a star or complete graph, as well as some study of coloring.

CHAPTER II

PRELIMINARIES

We begin with some background and notation both in semigroup theory and graph theory.

II.1 Semigroup Theory

First, a *semigroup* S is a nonempy set on which an associative binary operation is defined, which will be denoted throughout this paper as multiplication. If $xy = yx$ for all $x, y \in S$, S is a *commutative semigroup.* If there exists an element 1 of S such that $x1 = 1x = x$ for all $x \in S$, we say that 1 is an *identity* element of S, and that S is a *semigroup with identity.* A semigroup S has at most one such identity element. If S has no identity element, one can simply be adjoined to S if need be. We shall consistently use the notation S^1, where

$$S^1 = \begin{cases} S & \text{if S has an identity element.} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

If a semigroup S with at least two elements contains an element 0 such that $x0 = 0x = 0$ for all $x \in S$, we say that 0 is a *zero element* of S, and that S is a *semigroup with zero.* Once again a semigroup S can contain at most one zero element. A semigroup S is called *reduced* if it contains no nonzero nilpotent element, and S is a *null semigroup* if $xy = 0$ for all $x, y \in S$.

Throughout this paper, each semigroup S will be commutative and contain a zero element.

A nonempty subset $I \subseteq S$ is called a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a *two-sided ideal* or simply *ideal* if it is both a left and right ideal. Since our semigroups S are commutative, every ideal will be a two-sided ideal. If a is an element of the commutative semigroup S, the smallest ideal containing a is called the *principal ideal generated by a.* As in rings, this ideal will contain $aS = \{as \,|\, s \in S\}$, the set of multiples of a. Since this set will not necessarily contain a, we denote the *principal ideal generated by* a as $aS^1 = aS \cup \{a\}$. S itself is an ideal of S, as is $\{0\}$. The zero ideal will be denoted (0). Note also that the product, union, and intersection of ideals of S will again be an ideal of S, and that each nonzero ideal must necessarily be composed of a union of principal ideals. An ideal I of S will be called *0-minimal* if it is minimal in the set of nonzero ideals, that is, if it properly contains no nonzero ideal of S.

We will consider decomposing a commutative semigroup S into a union of subsets with certain properties. A semigroup S will be referred to as a *0-disjoint union* of subsets A and B if $S = A \cup B$ and $A \cap B = \{0\}$. A similar idea is that of an *irredundant union*. Recall that an ideal I of S is composed of a union of principal ideals. We will have cause to represent this union in the most concise way. An ideal I of a commutative semigroup S is represented as an *irredundant union* of principal ideals if $I = \bigcup\limits_{\alpha \in \Lambda} a_\alpha S^1$ but $I \neq \bigcup\limits_{\beta \in \Lambda'} a_\beta S^1$ for any proper subset $\Lambda' \subset \Lambda$. Note that the union $I = \bigcup\limits_{\alpha \in \Lambda} a_\alpha S^1$ is irredundant precisely when there are no containment relations between the $a_\alpha S^1$.

It will at times be useful to consider the quotient of a semigroup by an ideal. This concept does not translate directly from the usual notion in ring theory, but the *Rees homomorphism* in semigroup theory does correspond closely to this familiar idea. Throughout this paper, we will use this *Rees quotient semigroup*, denoted S/I where I is an ideal of S. Here, $S/I = (S \setminus I) \cup \{0\}$, with

$$xy = \begin{cases} xy & \text{if } xy \in S \setminus I \\ 0 & \text{if } xy \in I \end{cases}$$

for all nonzero $x, y \in S \setminus I$. For a complete discussion of the semigroup congruence behind this concept, see [12].

We will often consider the structure of a semigroup S in terms of *annihilators*. For a subset $A \subseteq S$, the *annihilator of* A, denoted $Ann(A)$ is given by $Ann(A) = \{x \in S \mid xa = 0 \text{ for all } a \in A\}$. We summarize common properties of the annihilator, which will be used without reference.

Lemma 1. *[2] If A and B are subsets of a commutative semigroup S, then*

1. *$Ann(A)$ is an ideal of S.*

2. *$Ann(0) = S$.*

3. *$A \subseteq B$ implies that $Ann(A) \supseteq Ann(B)$.*

4. *$A \subseteq Ann(Ann(A))$ and $Ann(A) = Ann(Ann(Ann(A)))$.*

II.2 Graph Theory

A *graph* G is an ordered pair $G = (V, E)$ where V, sometimes denoted $V(G)$, is a set of *vertices* and E, sometimes denoted $E(G)$, is a set of *edges*, which are 2-element subsets of $V(G)$. Two elements $x, y \in V(G)$ are *adjacent* if they are joined by an edge, that is, if the set $\{x, y\}$ is an element of $E(G)$. A *path* is a sequence of edges which connects a sequence of vertices. A path which starts and ends at the same vertex is called a *cycle*. For distinct vertices $x, y \in V(G)$, let the *distance* between x and y, denoted $d(x, y)$, be the length of the shortest path between them. We adopt the convention that if no such path exists, then $d(x, y) = \infty$. The *diameter* of the graph G, denoted $diam(G)$, is given by $diam(G) = \sup\{d(x, y) \mid x, y \in V(G), x \neq y\}$. The *girth* of a graph G, denoted $gr(G)$ is defined as the length of the shortest cycle in G, with $gr(G) = \infty$ if G contains no cycles.

If $x \in V(G)$, let $N(x)$ denote the *neighborhood* of x, defined as the set of vertices in G which are adjacent to x, and let $\overline{N(x)} = N(x) \cup \{x\}$. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*, and a complete graph on n vertices is denoted K_n. A *star graph* is a graph G which contains one vertex, called the *center*, to which all other vertices are joined, and no other edges.

Two graphs G and H are said to be *isomorphic*, denoted $G \cong H$, if there exists a structure-preserving bijection $f : V(G) \to V(H)$. If a graph G contains a subgraph which is isomorphic to K_n, the complete graph on n vertices, this subgraph is called a *clique* of G. The *clique number* of a graph G, denoted $\omega(G)$ is defined as the supremum of the sizes of cliques in G. If a graph G has no vertices, its clique number is defined to be 0.

A *vertex k-coloring* of the graph G is an assignment $f : V(G) \to \{1, 2, \ldots, k\}$, where the elements $1, 2, \ldots, k$ are referred to as *colors*. A vertex coloring is considered *proper* if no two adjacent vertices are assigned the same color, and a graph G is said to be *k-colorable* if there exists a proper vertex k-coloring of G. The *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colors for which a proper vertex coloring exists. Note that for any graph G, $\omega(G) \leq \chi(G)$.

A more complete treatment of these topics in graph theory can be found in [11].

Let $\mathbb{I}(S)$ be the set of ideals of a commutative semigroup S with zero. As in the ring case, we define a proper ideal $I \in \mathbb{I}(S)$ to be an *annihilating-ideal* of S if there exists a nonzero ideal $J \in \mathbb{I}(S)$ such that $IJ = (0)$. Let $\mathbb{A}(S)$ be the set of all annihilating-ideals of S, and $\mathbb{A}^*(S) = \mathbb{A}(S) \setminus \{(0)\}$. We then define the *annihilating-ideal graph*, $\mathbb{AG}(S)$, as a graph with vertex set $\mathbb{A}^*(S)$, in which distinct vertices I and J are adjacent if and only if $IJ = (0)$.

CHAPTER III

FINITENESS CONDITIONS FOR ANNIHILATING-IDEAL GRAPHS

A nonempty family of sets $\mathcal{R}$ is called a *ring of sets* if it contains the union and intersection of any two of its sets.

Lemma 2. *[5] Let $\mathcal{R}$ be a ring of sets consisting of subsets of a set A. If $\mathcal{R}$ is infinite, then $\mathcal{R}$ contains an infinite chain.*

Proof. Assuming the axiom of choice, $\mathcal{R}$ contains a maximal chain. Let $\mathcal{M}$ be a maximal chain in $\mathcal{R}$, and suppose $\mathcal{M}$ is finite. Then, $\mathcal{M}$ has the form

$$M_1 \subset M_2 \subset \cdots \subset M_n$$

for some $n \in \mathbb{N}$. If $M_1 \neq \emptyset$, set $M_0 = \emptyset$, and if $M_n \neq A$, set $M_{n+1} = A$. Now, from the chain

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_n \subset M_{n+1} = A$$

we can form a partition of the set A, with the difference sets $M_{i+1} \setminus M_i$ for $i = 0, 1, 2, \ldots n$, as the blocks of the partition. Since $\mathcal{R}$ is an infinite ring of sets, there will be some set $\mathcal{B} \in \mathcal{R}$ such that $\mathcal{B}$ is different from a union of such blocks. But, since $\mathcal{B} \subset A$ and these blocks partition A, there must be some $k \leq n$ such that $M_{k+1} \setminus M_k$ contains some elements of $\mathcal{B}$ and some elements which are not in $\mathcal{B}$. Then, $M_{k+1} \cap \mathcal{B} \neq \emptyset$, and since $\mathcal{R}$ is a ring of sets, $(M_{k+1} \cap \mathcal{B}) \cup M_k \in \mathcal{R}$ which can be properly inserted into the chain $\mathcal{M}$ between M_k and M_{k+1}. This, however contradicts the fact that $\mathcal{M}$ was chosen as a maximal chain. Thus, any maximal chain in $\mathcal{R}$ must be infinite, and so $\mathcal{R}$ contains an infinite chain. $\square$

Recall that the union and intersection of ideals of a semigroup S is again an ideal of S. Thus, $\mathbb{I}(S)$, the set of all ideals of S, is a ring of sets.

Theorem 3. *A commutative semigroup S satisfies ACC and DCC on its ideals if and only if S contains finitely many principal ideals.*

Proof. Suppose S contains finitely many principal ideals. Then, since each ideal I of S is some union of principal ideals, the set $\mathbb{I}(S)$ must also be finite, and so any chain of ideals in $\mathbb{I}(S)$ must be finite, giving that S ACC and DCC on its ideals. Conversely, if S satisfies ACC and DCC on its ideals, every chain of ideals in $\mathbb{I}(S)$ is finite. Since $\mathbb{I}(S)$ is a ring of sets that contains no infinite chains, $\mathbb{I}(S)$ must be finite by Lemma 2, and so S contains only finitely many ideals. Thus, there can only be finitely many principal ideals. $\square$

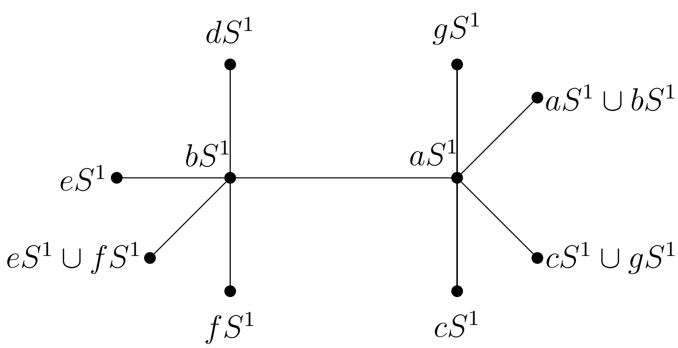
In order to restrict ourselves to finite annihilating-ideal graphs, we will consider only those semigroups with finitely many principal ideals. We will also note that each annihilating-ideal of a semigroup S will necessarily consist entirely of zero divisors of S, and therefore we will consider only commutative semigroups with zero which consist entirely of zero divisors. These will be referred to as *zero divisor semigroups*.

Commutative rings in which every nonzero proper ideal I of a ring R is a vertex in the graph $\mathbb{AG}(R)$ are abundant. In fact, for every Artinian ring R, every nonzero proper ideal represents a vertex in the graph. It has further been conjectured that for any non-domain ring R, the set of vertices of $\mathbb{AG}(R)$ and the set of nonzero proper ideals of R have the same cardinality. This is not the case for commutative zero divisor semigroups, as can be seen in the following example.

Example 4. Let S be the semigroup given by multiplication table:

·	0	a	b	c	d	e	f	g
0	0	0	0	0	0	0	0	0
a	0	0	0	0	a	a	a	0
b	0	0	b	b	0	0	0	b
c	0	0	b	b	a	a	a	b
d	0	a	0	a	d	d	d	a
e	0	a	0	a	d	e	d	a
f	0	a	0	a	d	d	f	a
g	0	0	b	b	a	a	a	b

This zero divisor semigroup contains nonzero, proper ideals which are *not* annihilating-ideals of S. For example, the ideal $bS^1 \cup eS^1$ is a nonzero, proper ideal of S, but $Ann(bS^1 \cup eS^1) = (0)$, and so $bS^1 \cup eS^1 \notin \mathbb{A}^*(S)$. It can be verified that $|\mathbb{I}^*(S)| = 24$, while $|\mathbb{A}^*(S)| = 10$, with $\mathbb{AG}(S)$:



Theorem 5 provides a sufficient condition for which every nonzero proper ideal of a commutative zero divisor semigroup S is an annihilating-ideal of S.

Theorem 5. *Let S be a commutative zero divisor semigroup. If $Ann(S) \neq (0)$, $\mathbb{A}^*(S) = \mathbb{I}^*(S)$.*

Proof. First, note that $Ann(S)$ is an ideal of S, since $Ann(S) \cdot S = 0 \in Ann(S)$. Then, since $Ann(S) \neq (0)$, $Ann(S) = S$ or $Ann(S) \in \mathbb{A}^*(S)$.

Let $I \in \mathbb{I}^*(S)$. Suppose $Ann(S) = S$. If $J \in \mathbb{A}^*(S)$ such that $J \neq I$, then $I \subset Ann(S)$, giving $IJ = (0)$ and so $I \in \mathbb{A}^*(S)$ as well.

Suppose $Ann(S) \in \mathbb{A}^*(S)$. Then, for any ideal $I \in \mathbb{I}^*(S)$, $I \cdot Ann(S) = (0)$, giving $I \in \mathbb{A}^*(S)$. Thus, in either case, $\mathbb{A}^*(S) = \mathbb{I}(S)$. □

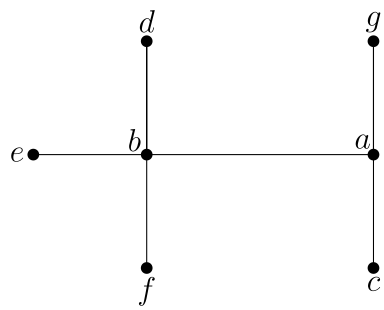
The converse of this theorem is not true in general, for example consider the zero divisor semigroup S given by $S = \{0, x, y \,|\, x^2 = x, y^2 = y, xy = 0\}$. For this S, $\mathbb{I}^*(S) = \{xS^1, yS^1\} = \mathbb{A}^*(S)$, while $Ann(S) = (0)$.

CHAPTER IV

ANNIHILATING-IDEAL GRAPHS AND ZERO-DIVISOR GRAPHS

Recall $\Gamma(S)$, the zero-divisor graph of S, in which nonzero zero divisors $x, y \in S$ with $x \neq y$ are connected by an edge in case $xy = 0$. We note that the annihilating-ideal graph $\mathbb{AG}(S)$ does in many cases provide different information than does $\Gamma(S)$.

Example 6. The zero-divisor graph of the semigroup S given in Example 4, $\Gamma(S)$:



Clearly in this example, $\mathbb{AG}(S)$ contains more structure than does $\Gamma(S)$. One reason for this is simply that there are more annihilating-ideals than there are elements in S. A perhaps more interesting contributing factor is the nilpotency of elements of S. The zero-divisor graph provides no information on the nilpotency of elements of S, as convention dictates that $\Gamma(S)$ remain a simple graph. Thus, if $x^2 = 0$ for some $x \in S$, $\Gamma(S)$ will not contain a loop at vertex x to illustrate this fact. While the annihilating-ideal graph is similarly taken to be a simple graph, the nilpotency of elements can be illustrated when a nilpotent element is contained in multiple annihilating-ideals, like the element a in the above semigroup.

Examples 4 and 6 also serve as an example that, in many cases, $\Gamma(S)$ can be found in $\mathbb{AG}(S)$ as a subgraph.

Theorem 7. *If every nonzero element of a commutative zero divisor semigroup S generates a unique, proper principal ideal, $\mathbb{AG}(S)$ will contain a subgraph which is isomorphic to $\Gamma(S)$, the zero-divisor graph of S.*

Proof. Let $a \in S$, $a \neq 0$. Then, $\exists b \in S$ such that $ab = 0$, $b \neq 0$. Then, $aS^1bS^1 = (0)$, giving that both $aS^1, bS^1 \in \mathbb{A}^*(S)$, and that aS^1, bS^1 are adjacent vertices in $\mathbb{AG}(S)$.
Now, consider the subgraph $Z\mathbb{G}(S)$ of $\mathbb{AG}(S)$ consisting of the vertices $\{aS^1 | a \in S, a \neq 0\} \subseteq \mathbb{A}^*(S)$ and edges between two vertices aS^1, bS^1 if and only if $ab = 0$. Then, the map from $Z\mathbb{G}(S) \to \Gamma(S)$ given by $aS^1 \mapsto a$ is a structure-preserving bijection, since vertices a, b in $\Gamma(S)$ are adjacent if and only if $ab = 0$. Thus, we have $Z\mathbb{G}(S) \cong \Gamma(S)$. □

The zero-divisor graph $\Gamma(S)$ of a commutative semigroup S is also useful in obtaining graph theoretic results for annihilating-ideal graphs, since the set of annihilating ideals, $\mathbb{A}(S)$ is itself a semigroup composed of zero divisors, and $\Gamma(\mathbb{A}(S)) = \mathbb{AG}(S)$.

Theorem 8. *For a commutative zero divisor semigroup S, the annihilating-ideal graph $\mathbb{AG}(S)$ satisfies all of the following conditions.*

1. *$\mathbb{AG}(S)$ is connected.*
2. *$diam(\mathbb{AG}(S)) \leq 3$.*
3. *If $\mathbb{AG}(S)$ contains a cycle, then the core of $\mathbb{AG}(S)$ is a union of quadrilaterals and triangles, and any vertex not in the core of $\mathbb{AG}(S)$ is an end, and so $gr(\mathbb{AG}(S) \leq 4$.*
4. *For each pair I, J of nonadjacent vertices of $\mathbb{AG}(S)$, there is a vertex L with $N(I) \cup N(J) \subset \overline{N(L)}$.*

Proof. All of these follow from treating $\mathbb{A}(S)$ as a semigroup and noting that $\Gamma(\mathbb{A}(S)) = \mathbb{AG}(S)$, and applying Theorem 1 in [9]. □

CHAPTER V

STAR GRAPHS AND COMPLETE GRAPHS

Theorem 5 indicates that the structure of $Ann(S)$ is useful in determining the structure of $\mathbb{AG}(S)$ for a given commutative zero divisor semigroup S. Since $Ann(S)$ is clearly an ideal of S, either $Ann(S) = (0)$, $Ann(S) = S$, or $Ann(S) \in \mathbb{A}^*(S)$. With this in mind, we now investigate semigroups whose annihilating-ideal graphs have a vertex adjacent to every other vertex, are star graphs, or are complete graphs.

Lemma 9. *Let S be a commutative zero divisor semigroup. If $Ann(S) = S$, $\mathbb{AG}(S)$ is a complete graph.*

Proof. Since $Ann(S) = S$, $\mathbb{I}^*(S) = \mathbb{A}^*(S)$, so let $I, J \in \mathbb{I}^*(S), I \neq J$. Then, $IJ \subseteq S^2 = S \cdot Ann(S) = (0)$, and so each vertex of $\mathbb{AG}(S)$ is adjacent to every other vertex. Thus, $\mathbb{AG}(S)$ is a complete graph. □

Lemma 10. *Let S be a commutative zero divisor semigroup. If $Ann(S) \in \mathbb{A}^*(S)$, there is a vertex of $\mathbb{AG}(S)$ which is adjacent to every other vertex.*

Proof. For each $I \in \mathbb{A}^*(S), I \neq Ann(S)$, $Ann(S) \cdot I = (0)$. Thus, $Ann(S)$ is a vertex of $\mathbb{AG}(S)$ which is adjacent to every other vertex. □

Lemma 11. *Let S be a commutative zero divisor semigroup. If $Ann(S) = (0)$, $xS^1 \in \mathbb{A}^*(S)$ for each nonzero $x \in S$.*

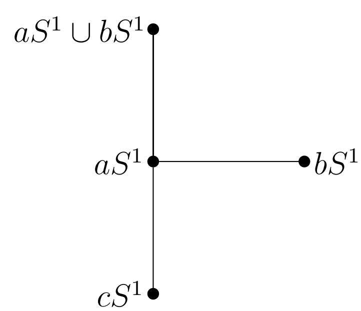
Proof. Let $x \in S$ such that $x \neq 0$. Since S is a zero divisor semigroup, $xy = 0$ for some nonzero $y \in S$, giving $y \in Ann(xS^1)$, so $Ann(xS^1) \neq (0)$. If $Ann(xS^1) = S$, $xS^1 = Ann(S)$, a contradiction. If $xS^1 = S$, $y \in Ann(S)$, a contradiction.Thus, $xS^1 \in \mathbb{A}^*(S)$. □

In several of the following results, we consider commutative zero divisor semigroups S which can be expressed as the 0-disjoint union of two ideals of S, as in the following example.

Example 12. Let S be the semigroup given by multiplication table:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	b
c	0	0	b	b

Note that we can express S as $S = aS^1 \cup cS^1$, where $as^1 \cap cS^1 = \{0\}$, aS^1 is a 0-minimal ideal of S, and if we consider $cS^1 = \{0, b, c\}$ as a semigroup in its own right, cS^1 contains no zero divisors. Further, viewing S as this 0-disjoint union of a 0-minimal ideal and another ideal gives that any ideal I of S must be equal to aS^1, entirely contained in cS^1, or of the form $aS^1 \cup J$, where $J \subseteq cS^1$. This ideal structure of S lends to the star shape of its annihilating-ideal graph, $\mathbb{AG}(S)$:



This exemplifies the following theorem.

Theorem 13. *Let S be a commutative zero divisor semigroup. If S is the 0-disjoint union of a 0-minimal ideal I and an ideal J which contains no zero divisors, then $\mathbb{AG}(S)$ is a star graph.*

Proof. First, note that $IJ \subseteq I$ gives $IJ = I$ or $IJ = (0)$, since I is a 0-minimal ideal of S. Since $IJ \subseteq J$ as well and $I \cap J = \{0\}$, it must be that $IJ = (0)$, giving I adjacent to J in $\mathbb{AG}(S)$.

Note that any ideal of S must be either I, an ideal K of J, or of the form $I \cup K$ for some nonzero ideal K of J.

Clearly, for an ideal K of J, $IK \subseteq IJ = (0)$, giving I adjacent to K in $\mathbb{AG}(S)$. And, since any other vertex $L \in \mathbb{A}^*(S)$ must contain elements from J, K cannot be adjacent to L, since J contains no zero divisors.

Suppose $I \cup K \in \mathbb{A}^*(S)$. Then, there exists $L \in \mathbb{A}^*(S)$ such that $(I \cup K)L = IL \cup KL = (0)$, giving $IL = KL = (0)$. Since J contains no zero divisors and $K \subseteq J$, it must be that $L \cap J = \{0\}$. Then, it can only be that $L = I$, and so an ideal of the form $I \cup K$ can only be adjacent to I in $\mathbb{AG}(S)$.

Thus, $\mathbb{AG}(S)$ is a star graph with center I. □

Theorem 14. *Let S be a commutative zero divisor semigroup. Then, there exists a vertex of $\mathbb{AG}(S)$ which is adjacent to every other vertex if and only if $Ann(S) \neq (0)$ or $\mathbb{AG}(S)$ is a star graph.*

Proof. ($\Leftarrow$) This direction follows from Lemma 9, Lemma 10, and the definition of a star graph.

($\Rightarrow$) Suppose there is a vertex I of $\mathbb{AG}(S)$ which is adjacent to every other vertex. If $Ann(S) \neq (0)$, there is nothing to prove.

Suppose then that $Ann(S) = (0)$, and let $a \in I, a \neq 0$. Then, $aS^1 \in \mathbb{A}^*(S)$, by Lemma 11, and so $aS^1 \subset I$ or $aS^1 = I$. If $aS^1 \subset I$, aS^1 is another vertex which is adjacent to every other vertex and $aS^1 \cdot I = (0)$, since I is adjacent to every vertex of $\mathbb{AG}(S)$. Then, $a^2 = 0$. Since $xS^1 \in \mathbb{A}^*(S)$ for each nonzero $x \in S$, we also have $aS^1 \cdot xS^1 = (0)$ and thus $ax = 0$ for each nonzero $x \in S$. Then, $a \in Ann(S)$, a contradiction. Thus, $aS^1 = I$ is our vertex which is adjacent to every other vertex.

We now show that aS^1 is a 0-minimal ideal of S. Suppose that $bS^1 \subset aS^1$ for some nonzero $b \in S, b \neq a$. Then, bS^1 must also be adjacent to every other vertex of $\mathbb{AG}(S)$. In particular, $bS^1 \cdot aS^1 = (0)$, giving $b^2 = 0$, again contradicting the fact that $Ann(S) = (0)$. Thus, aS^1 is a 0-minimal ideal of S.

Consider $aS^1 \cap Ann(aS^1)$. If $x \in aS^1 \cap Ann(aS^1)$ for some nonzero $x \in S$, $xS^1 \subseteq aS^1$, and so it must be that $xS^1 = aS^1$, since aS^1 is 0-minimal. Then, xS^1 is adjacent to every other vertex of $\mathbb{AG}(S)$ and $x \in Ann(aS^1)$ gives $x \in Ann(xS^1)$, and so $x^2 = 0$, again a contradiction. Thus, $aS^1 \cap Ann(aS^1) = \{0\}$.

Now, we claim $aS^1 \cup Ann(aS^1) = S$. If $x \in S$ such that $x \notin aS^1$, $xS^1 \in \mathbb{A}^*(S)$ that is distinct from aS^1. Then, xS^1 is adjacent to aS^1 in $\mathbb{AG}(S)$, giving $xS^1 \cdot aS^1 = (0)$, and thus $x \in Ann(aS^1)$.

Finally, suppose $Ann(aS^1)$ contains zero divisors, so there are nonzero $x, y \in Ann(aS^1)$ such that $xy = 0$. Then, consider the ideal $xS^1 \cup aS^1$, which is either equal to S, an element of $\mathbb{A}^*(S)$, or has $Ann(xS^1 \cup aS^1) = (0)$. If $(xS^1 \cup aS^1) \in \mathbb{A}^*(S)$, it must be adjacent to aS^1, giving $a^2 = 0$, a contradiction. And, $y \in Ann(xS^1 \cup aS^1)$, so $Ann(xS^1 \cup aS^1) \neq (0)$. Thus, it must be that $xS^1 \cup aS^1 = S$. But, that gives $Ann(S) = Ann(xS^1) \cap Ann(aS^1)$, and thus $y \in Ann(S)$, a contradiction as well. Thus, $Ann(aS^1)$ can contain no zero divisors.

Thus, we have that S is the 0-disjoint union of the 0-minimal ideal aS^1 and the ideal $Ann(aS^1)$ which contains no zero divisors, and so by Theorem 12, $\mathbb{AG}(S)$ is a star graph. □

Corollary 15. *Let S be a commutative zero divisor semigroup with $Ann(S) = (0)$. Then, $\mathbb{AG}(S)$ is a star graph if and only if S is the 0-disjoint union of a 0-minimal ideal I and $Ann(I)$, where $Ann(I)$ contains no zero divisors.*

Proof. This follows directly from Theorem 13 and the proof of Theorem 14. □

We now further investigate those commutative zero divisor semigroups S with $Ann(S) \neq (0)$ and whose annihilating-ideal graphs are star graphs. If we refer back to Example 12, we have in this example that $Ann(S) = aS^1$ and that the Rees quotient semigroup

$S/Ann(S) = (S \setminus Ann(S)) \cup \{0\} = cS^1$, and so this example also serves to illustrate the following lemma.

Lemma 16. *Let S be a commutative zero divisor semigroup such that $Ann(S) \in \mathbb{A}^*(S)$ and $\mathbb{AG}(S)$ is a star graph. Then, $S/Ann(S)$ is an ideal of S if and only if $S/Ann(S)$ contains no zero divisors.*

Proof. ($\Rightarrow$)First, note that $Ann(S)$ is adjacent to every other vertex of $\mathbb{AG}(S)$, and so $Ann(S)$ is the center of the star graph, and $Ann(S)$ must be a 0-minimal ideal of S. Suppose there exist $x, y \in S \setminus Ann(S)$ such that $xy = 0$ in the Rees quotient semigroup $S/Ann(S)$. Then, either $xy = 0$ in S, or $xy \in Ann(S)$.

If $xy = 0$ in S, then $y \in Ann(xS^1)$, and either $Ann(xS^1) = S$ or $Ann(xS^1) \in \mathbb{A}^*(S)$. If $Ann(xS^1) = S$, then $xS = \{0\}$, giving $x \in Ann(S)$, a contradiction since $x \in S \setminus Ann(S)$. If $Ann(xS^1) \in \mathbb{A}^*(S)$, xS^1 is adjacent to $Ann(xS^1)$, contradicting the star shape of $\mathbb{AG}(S)$. Thus, $xy \neq 0$ in S. Then, it must be that $xy \in Ann(S)$. But, since $S/Ann(S) = S \setminus Ann(S) \cup \{0\}$ is an ideal of S, and $xy \neq 0$ in S, it must be that $xy \in S \setminus Ann(S)$, a contradiction. Thus, there cannot be zero divisors in $S/Ann(S)$.

($\Leftarrow$)Suppose the Rees quotient semigroup $S/Ann(S) = (S \setminus Ann(S)) \cup \{0\}$ contains no zero divisors. Consider then $S/Ann(S) \cdot S = (S \setminus Ann(S) \cup \{0\}) \cdot Ann(S) \cup (S \setminus Ann(S) \cup \{0\}) \cdot S \setminus Ann(S) = (S \setminus Ann(S))^2 \cup \{0\}$. Let $x, y \in S \setminus Ann(S)$. Since $xy \neq 0$ in $S/Ann(S)$, $xy \notin Ann(S)$, giving $xy \in S \setminus Ann(S)$. Thus, $(S \setminus Ann(S))^2 \subseteq S \setminus Ann(S)$, giving $S/Ann(S) \cdot S \subseteq S/Ann(S)$, and so $S/Ann(S)$ is an ideal of S. $\square$

However, this is not the only situation in which $Ann(S) \neq (0)$ and $\mathbb{AG}(S)$ is a star graph. To further categorize commutative zero divisor semigroups whose annihilating-ideal graphs are star graphs, we will make use of the following lemmas.

Lemma 17. *Let S be a commutative zero divisor semigroup such that $Ann(S) \in \mathbb{A}^*(S)$ and $\mathbb{AG}(S)$ is a star graph. Then, $Ann(S)$ is the unique 0-minimal ideal of S if and only if $S/Ann(S)$ is not an ideal of S.*

Proof. ($\Rightarrow$)Suppose $Ann(S)$ is the unique 0-minimal ideal of S. Then, $Ann(S)$ is contained in every other nonzero ideal of S, and so the Rees quotient semigroup $S/Ann(S) = S \setminus Ann(S) \cup \{0\}$ cannot be an ideal of S.

($\Leftarrow$) Suppose $S/Ann(S)$ is not an ideal of S. Then, there exist $a \in S \setminus Ann(S)$ and $b \in S$ such that $ab \in Ann(S)$, $ab \neq 0$. Note that since $\mathbb{AG}(S)$ is a star graph with $Ann(S) \in \mathbb{A}^*(S)$, $Ann(S)$ must be the vertex at the center of $\mathbb{AG}(S)$. Since $Ann(S)$ is 0-minimal, it must be a principla ideal, say $Ann(S) = xS^1$ for some $x \in S$. Further, since $x \in Ann(S)$, we have $xS = \{0\}$, so it must be that $Ann(S)$ is of the form $xS^1 = \{0, x\}$. Thus, since $ab \in Ann(S)$ with $ab \neq 0$, it must be that $ab = x$.

Suppose yS^1 is another 0-minimal ideal of S for some nonzero $y \in S$. Consider then $aS^1 \cdot yS^1$. Since $aS^1 \cdot yS^1 \subseteq yS^1$, it must be that $aS^1 \cdot yS^1 = (0)$ or $aS^1 \cdot yS^1 = yS^1$, by 0-minimality. If $aS^1 \cdot yS^1 = (0)$, it must be that $yS^1 = Ann(S)$, by the star shape of $\mathbb{AG}(S)$. Else, $aS^1 \cdot yS^1 = yS^1$, giving that $yS^1 \subseteq aS^1$. Then, $y = ac$ for some nonzero $c \in S^1$. Then, $yS^1 \cdot bS^1 = acS^1 \cdot bS^1 = cS^1 \cdot abS^1 = cS^1 \cdot xS^1 = (0)$, and so yS^1, bS^1 are adjacent in $\mathbb{AG}(S)$, again giving that $yS^1 = Ann(S)$.

Thus, any 0-minimal ideal of S must in fact be equal to $Ann(S)$, so $Ann(S)$ is the unique 0-minimal ideal of S. □

Recall that each ideal I of a commutative semigroup S is the union of principal ideals of S. Since we are restricting ourselves to commutative zero divisor semigroups S with finitely many ideals, each ideal I of S can be expressed as a finite irredundant union of principal ideals.

Lemma 18. *Let S be a commutative zero divisor semigroup, and let I be an ideal of S such that $I = \bigcup_{i=1}^{n} a_iS^1$ is an irredundant union representation of I. If $IS = I$, $a_iS^1 \cdot S = a_iS^1$ for each $i = 1, \ldots, n$.*

Proof. We proceed by induction. Suppose $I = a_1S^1$ for some $a_1 \in S$. Then, $IS = S$ gives $a_1S^1 \cdot S = a_1S^1$. Suppose the claim holds for any ideal which is an irredundant union of $k-1$

principal ideals for some k. Then, let $I = \bigcup_{i=1}^{k} a_i S^1$ be an irredundant union of k principal ideals. Then, $I = J \cup a_k S^1$, where $J = \bigcup_{i=1}^{k-1} a_i S^1$. Then, $IS = (J \cup a_k S^1)S = JS \cup a_k S^1 \cdot S = a_1 S^1 \cdot S \cup a_2 S^1 \cdot S \cup \cdots \cup a_k S^1 \cdot S$. Note that J is an ideal of S which is an irredundant union of $k-1$ principal ideals, so our induction hypothesis gives that $a_i S^1 \cdot S = a_i S^1$ for each $i = 1, \ldots, k-1$. Then, since $IS = I = a_1 S^1 \cup a_2 S^1 \cup \cdots \cup a_k S^1$, it must be that $a_k S^1 \cdot S = a_k S^1$ as well. Thus, the claim holds for an irredundant union of k principal ideals as well. □

Theorem 19. *Let S be a commutative zero divisor semigroup such that $Ann(S) \neq (0)$ and $\mathbb{AG}(S)$ is a star graph. Then, either $S/Ann(S)$ is an ideal of S, or $S^4 = (0)$.*

Proof. First, if $Ann(S) = S$, $S^2 = (0)$, giving $S^4 = (0)$.

Suppose $Ann(S) \in \mathbb{A}^*(S)$. Then, $Ann(S)$ must be the center of the star graph $\mathbb{AG}(S)$, and must also be a 0-minimal ideal of S, so $Ann(S) = xS^1 = \{0, x\}$ for some $x \in S$. Consider then the Rees quotient semigroup, $S/Ann(S) = S \setminus Ann(S) \cup \{0\}$, which may or may not be an ideal of S. If $S/Ann(S)$ is an ideal of S, we are done.

Suppose $S/Ann(S)$ is not an ideal of S. Then, by Lemma 17, $Ann(S)$ is the unique 0-minimal ideal of S. Consider the chain of ideals

$$S \supseteq S^2 \supseteq S^3 \supseteq \cdots \supseteq (0).$$

Since S satisfies ACC and DCC on ideals, this chain must either stabilize at $S^n = S^{n+1} = \cdots$ for some n, or at $S^n = (0)$ for some n. Suppose $S^n = S^{n+1} = \cdots$ for some n. Then, let $S^n = \bigcup_{i=1}^{k} a_i S^1$ be an irredundant union representation of S^n. Since $S^n = S^{n+1} = S^n S$, Lemma 19 gives $a_i S^1 \cdot S = a_i S^1$ for each $i = 1, \ldots, k$. Now, since $Ann(S) = xS^1$ is the unique 0-minimal ideal of S, $xS^1 \subseteq S^n = \bigcup_{i=1}^{k} a_i S^1$. Then, $x \in a_j S^1$ for some j, giving $x = a_j t$ for some $t \in S^1$. And, since $a_j S^1 = a_j S^1 \cdot S$, $a_j = a_j uv$ for some $u \in S^1, v \in S$. Then, we have

$$x = a_j t = (a_j uv)t = (a_j t)(uv) = x(uv) = 0,$$

since $x \in Ann(S)$ and $uv \in S$, a contradiction.

Thus, it must be that if $S/Ann(S)$ is not an ideal of S, $S^n = (0)$ for some n. Take n to be the smallest such integer with $S^n = (0)$ and $S^{n-1} \neq (0)$. For any $I \in \mathbb{A}^*(S)$, we have $S^{n-1}I \subseteq S^{n-1}S = S^n = (0)$, giving that $S^{n-1} = Ann(S) = xS^1$, the center of the star graph $\mathbb{AG}(S)$.

Suppose $n > 4$. Then, S^2 and S^{n-2} will be distinct ideals of S and elements of $\mathbb{A}^*(S)$, and $S^n S^{n-2} = S^n = (0)$, giving that S^2 and S^{n-2} are adjacent in $\mathbb{AG}(S)$, a contradiction to the star shaped assumption. Thus, $n \leq 4$, and so $S^4 = (0)$. □

We are now in a position to completely characterize commutative zero divisor semigroups S with star-shaped annihilating-ideal graphs.

Theorem 20. *Let S be a commutative zero divisor semigroup. Then, $\mathbb{AG}(S)$ is a star graph if and only if either $Ann(S) = (0)$ and S is the 0-disjoint union of a 0-minimal ideal I and $Ann(I)$, where $Ann(I)$ contains no zero divisors, or $Ann(S) \neq (0)$ and one of the following holds:*

1. *$S/Ann(S)$ is an ideal of S.*

2. *$S^2 = (0)$ and $|\mathbb{I}^*(S)| = 1$ or 2.*

3. *$S^3 = (0)$, S^2 is the unique 0-minimal ideal of S, and if $I, J \in \mathbb{A}^*(S)$ such that $I \neq J$, $I, J \neq S^2$, then $IJ = S^2$.*

4. *$S^4 = (0)$, S^3 is the unique 0-minimal ideal of S, and if $I, J \in \mathbb{A}^*(S)$ such that $I \neq J$, $I, J \neq S^2$, $I, J \neq S^3$, then $IS^2 = JS^2 = S^3$ and $IJ = S^2$.*

Proof. (⇒) Suppose $\mathbb{AG}(S)$ is a star graph. If $Ann(S) = (0)$, Corollary 15 gives that S is the 0-disjoint union of a 0-minimal ideal I and $Ann(I)$, where $Ann(I)$ contains no zero divisors.

Suppose $Ann(S) \neq (0)$. By Theorem 20, $S/Ann(S)$ is an ideal of S, or $S^4 = (0)$. If $S/Ann(S)$ is an ideal of S, we are done. Else, we proceed by cases:

Case 1 Suppose $S^2 = (0)$. Then, by Lemma 9, $\mathbb{AG}(S)$ is a complete graph, and since it is also a star graph, it must be that $|\mathbb{A}^*(S)| = 1$ or 2. And, by Theorem 5, $\mathbb{A}^*(S) = \mathbb{I}^*(S)$ when $Ann(S) \neq (0)$, giving $|\mathbb{A}^*(S)| = |\mathbb{I}^*(S)| = 1$ or 2.

Case 2 Suppose $S^3 = (0)$ while $S^2 \neq (0)$. Then, $S^2 S = S^3 = (0)$, giving $S^2 \subseteq Ann(S)$. Since $S^2 \neq (0)$, we know that $Ann(S) \neq S$, and so $Ann(S) \in \mathbb{A}^*(S)$. Since $Ann(S)$ will necessarily be adjacent to every other element of $\mathbb{A}^*(S)$ and $\mathbb{AG}(S)$ is a star graph, $Ann(S)$ must be the 0-minimal ideal at the center of the star graph, so $Ann(S) = aS^1$ for some nonzero $a \in S$. Then, since S^2 is a nonzero ideal contained in $Ann(S)$, it must be that $S^2 = Ann(S) = aS^1$.

Suppose bS^1 is another 0-minimal ideal of S. Note that any two distinct 0-minimal ideals are necessarily 0-disjoint. Then, the ideal $bS^1 \cup aS^1 = S$, or $bS^1 \cup aS^1 \in \mathbb{A}^*(S)$. If $bS^1 \cup aS^1 = S$, $S/Ann(S) = S \setminus Ann(S) \cup \{0\} = bS^1$, and so we are in the case that $S/Ann(S)$ is an ideal of S. Else, suppose $bS^1 \cup aS^1 \in \mathbb{A}^*(S)$. We show $b^2 \neq 0$: Suppose $b^2 = 0$. Then, $(bS^1 \cup aS^1) \cdot bS^1 = (0)$, giving $bS^1 \cup aS^1$ adjacent to bS^1, a contradiction to the star shape of $\mathbb{AG}(S)$. Thus, $b^2 \neq 0$. Then, since $b^2 \neq 0$ and $b^2 \in bS^1$, a 0-minimal ideal, it must be that $b^2 S^1 = bS^1$. But, $b^2 \in S^2 = Ann(S) = aS^1$ as well, and so again since $b^2 \neq 0$ and aS^1 is a 0-minimal ideal, it must be that $aS^1 = b^2 S^1 = bS^1$. Thus, $aS^1 = Ann(S)$ is the unique 0-minimal ideal of S.

Now, let $I, J \in \mathbb{A}^*(S)$ such that $I \neq J$, $I, J \neq S^2$. Then, $IJ \subseteq S^2$, and since $S^2 = Ann(S)$ is a 0-minimal ideal of S, it must be that $IJ = (0)$ or $IJ = S^2$. By the star shape of $\mathbb{AG}(S)$, I and J cannot be adjacent, and so $IJ = S^2$.

<u>*Case 3*</u> Suppose $S^4 = (0)$ while $S^3 \neq (0)$. Then, by similar reasoning to Case 2, it must be that $S^3 = Ann(S)$ is the unique 0-minimal ideal of S which is the center of the star graph $\mathbb{AG}(S)$. Then, S^2, S^3 are distinct elements of $\mathbb{A}^*(S)$, with $S^3 \subset S^2$.

We now show that the chain of ideals $S^2 \supset S^3 \supset S^4 = (0)$ is a maximal chain. Since, $S^3 \subset S^2$, there exist elements $x, y \in S \setminus S^2$ such that $xy \in S^2 \setminus S^3$. Then, $xyS^1 \cdot S^2 \subseteq S^4 = (0)$. By the star shape of $\mathbb{AG}(S)$, it must be that either $xyS^1 = S$, $xyS^1 = S^3$, or $xyS^1 = S^2$. If $xyS^1 = S$, $S^3 = (0)$, a contradiction. If $xyS^1 = S^3$, then $xy \in S^3$, a contradiction since $xy \in S^2 \setminus S^3$. Thus, it must be that $xyS^1 = S^2$.

Suppose $x^2 \in S^3$. Then, $xS^1 \cdot xyS^1 = x^2S^1 \cdot yS^1 = (0)$, since $x^2 \in S^3 = Ann(S)$. The star shape of $\mathbb{AG}(S)$ again gives $xS^1 = S$, $xS^1 = S^2$, or $xS^1 = S^3$, since under no other circumstances can xS^1 and $xyS^1 = S^2$ be adjacent. If $xS^1 = S$, then $S^3 = (0)$, a contradiction. If $xS^1 = S^2$, then $x \in S^2$, a contradiction since $x \in S \setminus S^2$. And, if $xS^1 = S^3$, then $xy \in S^3$, a contradiction. Thus, it cannot be that $x^2 \in S^3$, and so repeating the previous argument with $x, x \in S \setminus S^2$ yields $x^2S^1 = S^2$. Similar star shape argument gives that x^3 cannot be equal to 0, and so $x^3S^1 \subseteq S^3$ is nonzero. Since S^3 is a 0-minimal ideal, it must be that $x^3S^1 = S^3$, and then $x^4S^1 = S^4 = (0)$. Thus, the chain $S^2 \supset S^3 \supset S^4$ can be expressed as $x^2S^1 \supset x^3S^1 \supset x^4S^1$ and must be a maximal chain.

Let $I \in \mathbb{A}^*(S), I \neq S^2, S^3$. Then, $IS^2 \subseteq S^3$, and since S^3 is 0-minimal, it must be that $IS^2 = (0)$ or $IS^2 = S^3$. By the star shape of $\mathbb{AG}(S)$, it must be that $IS^2 = S^3$ for all such I. Let $J \in \mathbb{A}^*(S)$ such that $J \neq I, S^2, S^3$. Then, $IJ \subseteq S^2$, and since $S^2 \supset S^3 \supset S^4$ is a maximal chain, it must be that $IJ = S^2$, $IJ = S^3$, or $IJ = (0)$. By the star shape of $\mathbb{AG}(S)$, $IJ \neq (0)$. If $IJ = S^3$, one of I, J must be an ideal entirely contained in S^2, a contradiction since S^3 is the only such ideal. Thus, it must be that $IJ = S^2$.

($\Leftarrow$) It follows directly that each of these cases yields a star-shaped $\mathbb{AG}(S)$.

$\square$

CHAPTER VI

COLORING OF ANNIHILATING-IDEAL GRAPHS

When Beck first assigned a graph whose vertices correspond to elements of a commutative ring R, his main interest was in the coloring of that graph. Denoting this graph also by R, Beck conjectured that $\chi(R) = \omega(R)$ for all commutative rings R. Beck was able to show that this held for reduced rings and principal ideal rings, and this has also been shown for the $\Gamma(S)$ of a reduced or idempotent semigroup S. But, in [4], Anderson and Naseer provided a counterexample to Beck's conjecture for rings.

Behboodi and Rakeei continued on this track with $\mathbb{AG}(R)$ in [7], [8]. They were able to establish the equality $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$ held for reduced rings, as well as for Anderson and Naseer's counterexample to Beck's conjecture. These results and a lack of counterexamples prompted them to conjecture that $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$ for every commutative ring R.

We show that similar equality holds for reduced semigroups as well as semigroups S with $Ann(S) = S$.

Lemma 21. *Let S be a commutative zero divisor semigroup. If S is a reduced semigroup, then $\mathbb{A}(S)$ is a reduced semigroup.*

Proof. Suppose aS^1 is a nonzero nilpotent element of $\mathbb{A}(S)$ for some nonzero $a \in S$. Then, $(aS^1)^n = a^n S^1 = (0)$ for some n. Then, $a^n = 0$, giving that a is a nonzero nilpotent element of S, a contradiction, since S is reduced. Thus, $\mathbb{A}(S)$ is a reduced semigroup as well. □

Theorem 22. *Let S be a commutative zero divisor semigroup. If S is a reduced semigroup, $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S))$.*

Proof. By Lemma 21, $\mathbb{A}(S)$ is a reduced semigroup. Since $\Gamma(\mathbb{A}(S)) = \mathbb{AG}(S)$ and Theorems 3.2, 3.11, and 4.3 in [13] give that $\chi(\Gamma(\mathbb{A}(S))) = \omega(\Gamma(\mathbb{A}(S)))$, for a reduced semigroup, we have that $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S))$. □

Theorem 23. *Let S be a commutative zero divisor semigroup. If $Ann(S) = S$, then* $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S)) = |\mathbb{A}^*(S)| = |\mathbb{I}^*(S)|$.

Proof. By Lemma 9 and Theorem 5, S is a complete graph with $|\mathbb{A}^*(S)| = |\mathbb{I}^*(S)|$ vertices. Then, $\mathbb{AG}(S)$ itself is a maximal clique, and so $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S)) = |\mathbb{A}^*(S)| = |\mathbb{I}^*(S)|$.

□

We continue by attempting to bound $\chi(\mathbb{AG}(S))$.

Theorem 24. *Let S be a commutative zero divisor semigroup. Then, the number of 0-minimal ideals of S gives a lower bound to $\chi(\mathbb{AG}(S))$.*

Proof. Let $M(S)$ be the set of 0-minimal ideals of S, and $I, J \in M(S)$, $I \neq J$. Then, $IJ \subseteq I$ and since I is a 0-minimal ideal, $IJ = (0)$ or $IJ = I$. And, since $IJ \subseteq J$ as well, it must be that $IJ = (0)$. Thus, each 0-minimal ideal of S is adjacent to every other 0-minimal ideal of S, giving that $M(S)$ forms a clique of $\mathbb{AG}(S)$. Thus, we have

$$|M(S)| \leq \omega(\mathbb{AG}(S)) \leq \chi(\mathbb{AG}(S)).$$

□

If $Ann(S) \in \mathbb{A}^*(S)$, another clique can be found by considering the subsets of $Ann(S)$. Let $\mathcal{P}(A)$ denote the *power set* of the set A, and $\mathcal{P}(A)^*$ the *power set* of A without the empty set.

Lemma 25. *Let S be a commutative zero divisor semigroup such that $Ann(S) \in \mathbb{A}^*(S)$. Then, $\mathcal{P}(Ann(S) \setminus \{0\})^*$ corresponds to a clique of $\mathbb{AG}(S)$.*

Proof. Let $A, B \in \mathcal{P}(Ann(S) \setminus \{0\})^*$, $A \neq B$. Then, $A \cup \{0\}, B \cup \{0\} \in \mathbb{A}^*(S)$. And, since $A \cup \{0\}, B \cup \{0\} \subseteq Ann(S)$, $A \cup \{0\} \cdot B \cup \{0\} = (0)$. Thus, each element of $\mathcal{P}(Ann(S) \setminus \{0\})^*$ with $\{0\}$ adjoined is adjacent to every other element of $\mathcal{P}(Ann(S) \setminus \{0\})^*$ with $\{0\}$ adjoined, giving that $\mathcal{P}(Ann(S) \setminus \{0\})^*$ corresponds to a clique of $\mathbb{AG}(S)$. □

Theorem 26. *Let S be a commutative zero divisor semigroup such that $Ann(S) \in \mathbb{A}^*(S)$. Then,* $2^{|Ann(S)\setminus\{0\}|} - 1 \le \chi(\mathbb{AG}(S))$.

Proof. By Lemma 25, $\mathcal{P}(Ann(S) \setminus \{0\})^*$ corresponds to a clique of $\mathbb{AG}(S)$. Thus,

$$|\mathcal{P}(Ann(S) \setminus \{0\})^*| = |\mathcal{P}(Ann(S) \setminus \{0\})| - 1$$
$$= 2^{|Ann(S)\setminus\{0\}|} - 1 \le \omega(\mathbb{AG}(S)) \le \chi(\mathbb{AG}(S)).$$

□

The following example shows a semigroup S for which this bound is tighter than that given in Theorem 24.

Example 27. Let S be the semigroup given by multiplication table:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	a

This semigroup S has 2 0-minimal ideals, aS^1 and bS^1, so Theorem 24 gives $2 \le \chi(\mathbb{AG}(S))$. But, $Ann(S) = aS^1 \cup bS^1 = \{0, a, b\} \in \mathbb{AG}(S)$, and so Theorem 26 gives a tighter bound, $2^2 - 1 = 3 \le \chi(\mathbb{AG}(S))$. This bound is still imperfect, however, as $\mathbb{AG}(S)$ is a complete graph with four vertices in this case, giving $\chi(\mathbb{AG}(S)) = 4$.

Using the zero-divisor graph $\Gamma(S)$, we can also obtain an upper bound for $\chi(\mathbb{AG}(S))$.

Theorem 28. *Let S be a commutative zero divisor semigroup and $\chi(\Gamma(S)) < \infty$. Then,* $\chi(\mathbb{AG}(S)) \le 2^{\chi(\Gamma(S))} - 1$.

Proof. Suppose $\chi(\Gamma(S)) = k$ and that $c : V(\Gamma(S)) \to \{1, \ldots, k\}$ is a proper vertex k-coloring of $\Gamma(S)$. Let $A_i = c^{-1}(\{i\})$, the set of vertices of $\Gamma(S)$ which are assigned color i under c, for each $i = 1, \ldots, k$.

Define a new map, $f : \mathbb{A}^*(S) \to \mathcal{P}(\{1, \ldots, k\})^*$, given by $f(I) = \{i \mid 1 \leq i \leq k,\ I \cap A_i \neq \emptyset\}$ for each $I \in \mathbb{A}^*(S)$. Then, each annihilating-ideal I maps to the set of colors that its elements have in $\Gamma(S)$ under c. We show that f is a proper vertex coloring of $\mathbb{AG}(S)$.

Let $I, J \in \mathbb{AG}(S)$, $I \neq J$, such that $IJ = (0)$, giving I adjacent to J. Suppose that these vertices are identically mapped under f, so $f(I) = f(J)$, $\{i \mid 1 \leq i \leq k,\ I \cap A_i \neq \emptyset\} = \{i \mid 1 \leq i \leq k,\ J \cap A_i \neq \emptyset\}$. This implies that $I \cap A_i = J \cap A_i$ for each $i \in f(I) = f(J)$: Let $x \in I \cap A_i$ for some i. Then, $i \in f(I) = f(J)$, so there exists $y \in J$ such that $c(y) = i$, giving $y \in J \cap A_i$. Consider xy. Since $x \in I$ and $y \in J$ it must be that $xy = 0$, since $IJ = (0)$. If x and y are distinct, $xy = 0$ gives that x and y are adjacent vertices in $\Gamma(S)$, a contradiction since $c(x) = c(y) = i$ and c is a proper vertex coloring. Thus, it must be that $x = y$, giving $I \cap A_i \subseteq J \cap A_i$. Similarly, $J \cap A_i \subseteq I \cap A_i$, and so these sets are equal for each $i \in f(I) = f(J)$.

Now, consider $\{1, \ldots, k\} \setminus f(I)$, the set of of the k colors of $\Gamma(S)$ which do not have a representative in the ideal I. Let $j \in \{1, \ldots, k\} \setminus f(I)$, so no element of I is mapped to j under c. Then, $I \cap A_j = J \cap A_j = \emptyset$. Since c is a proper vertex coloring, the sets $A_1, \ldots, A_k$ partition the set $S \setminus \{0\}$, we deduce that $I = J$, a contradiction since I and J are assumed to be distinct ideals.

Thus, f is a proper vertex coloring of $\mathbb{AG}(S)$, and so

$$\chi(\mathbb{AG}(S)) \leq |\mathcal{P}(\{1, \ldots, k\})^*| = 2^{\chi(\Gamma(S))} - 1.$$

□

In our investigation of the coloring of $\mathbb{AG}(S)$, we have found no commutative semigroups such that $\chi(\mathbb{AG}(S)) \neq \omega(\mathbb{AG}(S))$, just as has been observed in the ring case. This motivates the following conjecture.

Conjecture 29. *For every commutative semigroup S, $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S))$.*

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