CALCULATION OF THE DECAY RATE FOR THE NEUTRINOLESS DOUBLE BETA DECAY PROCESS

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I dedicate this work to my thesis supervisor Dr. Mihai Horoi and Dr. Roman Senkov, who have supported me throughout this project and without their help it would be quite impossible for me to accomplish this formidable task.

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ABSTRACT

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by Shiplu Sarker

There has been a growing interest in neutrinoless double beta decay in recent years in the search for new physics beyond the Standard Model of particle physics. Since the process is sensitive to lepton number violation predicted by some gauge theories of physics beyond the Standard Model, it is expected to provide important information on whether neutrinos are Majorana or Dirac fermions and about the absolute neutrino mass-scale and the neutrino mass hierarchy. To achieve these goals one needs an accurate evaluation of the nuclear matrix element that enters in the half-life formula of the decay. In this project we include relativistic effects and higher order terms in the nucleon current for the correct evaluation of the matrix element to calculate the decay rate.

TABLE OF CONTENTS

LIST OF TABLES

TABLE PAGE

LIST OF FIGURES

FIGURE PAGE

CHAPTER I

OUTLINE OF THE THESIS

The purpose of this project is to find an expression for the decay rate of the neutrinoless double beta ($0\nu\beta\beta$) decay process, $\Gamma^{0\nu}$, which is related to the inverse half-life, $T_{1/2}^{0\nu}$, by:

$$\frac{\Gamma^{0\nu}}{ln2} = \left[T_{1/2}^{0\nu}\right]^{-1} = G_1^{0\nu}|M^{0\nu}|^2 \left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2. \tag{1.1}$$

Here $G_1^{0\nu}$ is the phase space factor, $M^{0\nu}$ is the nuclear matrix element, $\langle m_{\beta\beta}\rangle$ is the effective neutrino mass and m_e is the mass of electron. This research project is organized as follows. The history of double beta decay is briefly reviewed in chapter II. Chapter III provides a brief exposure of our current knowledge about neutrino masses and mixings in connection with the $0\nu\beta\beta$ decay, ordinary β decay and cosmology, since the effective neutrino mass, $\langle m_{\beta\beta}\rangle$, is related to the neutrino mass eigenvalues, m_k, through oscillation parameters. Chapter IV provides the necessary Quantum field theoretical background for the calculation of the decay rate. The main objective of this thesis is to include relativistic effects and higher order terms in the nucleon current, $J_L^{\mu\dagger}(x)$ for the correct evaluation of the nuclear matrix elements and hence to calculate the decay rate, which is given in chapter V. In chapter VI, we have presented some numerical results of the phase space factors, calculated by different authors. The conclusion of the thesis is given in chapter VII. The details of these calculations can be found in the appendices.

CHAPTER II

A BRIEF HISTORY OF DOUBLE BETA DECAY

2.1. The early period

The idea of double beta decay was first proposed by Maria Goeppert-Mayer [1] in 1935, shortly after Fermi's theory (1934) of β decay appeared. In the work of Maria Goeppert-Mayer, an expression for the two-neutrino double beta $(2\nu\beta\beta)$ decay rate, $\Gamma^{2\nu}$, was derived and a half-life, $T_{1/2}^{2\nu}$ of 10^{17} years was estimated by assuming a Q value of about 10 MeV. In 1937, Ettore Majorana theoretically demonstrated that all results of β decay theory remain unchanged if the neutrino, ν and the antineutrino, $\bar{\nu}$ are indistinguishable, i.e. if neutrino is a Majorana particle, and suggested antineutrino induced β^- decay for experimental verification of this hypothesis [2].

In 1939, Wolfgang Furry [3] for the first time proposed that, if neutrino is a Majorana particle, double beta decay can proceed without emission of any neutrino at all; this process which is now called the neutrinoless double beta $(0\nu\beta\beta)$ decay:

$$(A, Z) \rightarrow (A, Z+2) + e^- + e^-. \qquad (2.1.1)$$

Furry also calculated the approximate decay rate for the $0\nu\beta\beta$ decay process.

In 1930-40s parity violation in weak interactions was not known and consequently calculations showed that $0\nu\beta\beta$ decay mode should be much more likely to occur than ordinary $2\nu\beta\beta$ decay mode (if neutrinos are Majorana particles). It was believed that due to a considerable phase-space advantage, the $0\nu\beta\beta$ decay mode dominated the double beta decay rate. The predicted half-lives of $0\nu\beta\beta$ decay, $T_{1/2}^{0\nu}$, were on the order of 10^{12} - 10^{15} years [4]. Later it was found that the measured lower limit on the double beta decay half-life far exceeds the predicted value [4]. Figure 1 shows a generic level diagram for the double beta decay process, which represents both the $0\nu\beta\beta$ decay and $2\nu\beta\beta$ decay modes.

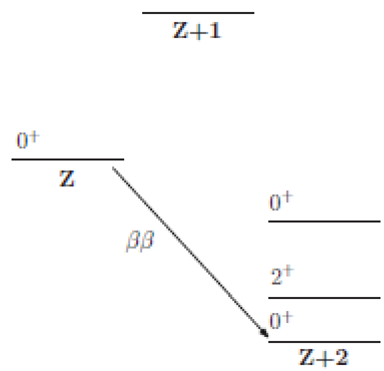

Figure 1. A generic level diagram for double beta decay.
Taken from Avignone F. T. *et al*, *Rev. Mod. phys*. **80**, 481 (2008).

In 1955, the Raymond Davis experiment [5], which searched for antineutrinos from the reactor via nuclear reaction, $\overline{\nu}_e + {}^{37}_{17}Cl \rightarrow {}^{37}_{18}Ar + e^-$, produced a zero result. The above experiments were interpreted as proof that the neutrino was not a Majorana particle, but a Dirac particle. This prompted the introduction of the lepton number (LN) to distinguish the neutrino from its antiparticle. The assumption of LN conservation allows the $2\nu\beta\beta$ decay process but forbids the $0\nu\beta\beta$ decay process, in which LN is changed by 2 units.

Efforts to observe the double beta decay process in laboratory date back to at least 1949, when Edward L. Fireman made the first attempt to measure the half-life of the Sn-124 isotope [6], but he disclaimed it later [7]. Radiometric experiments through about 1960 produced either negative results or false positives, which were not confirmed by later experiments [8]. In 1950, for the first time, the half-life of the Te-130 isotope was measured by geochemical methods with an estimated half-life of 1.4×10^{21} years [9], which is reasonably close to the modern value [8]. Extensive studies have been made by Gentner and Kirsten [10, 11] and others [12, 13] on such rare-gas isotopes as Kr-82, Xe-128 and Xe-130, which are $\beta\beta$ decay products of Se-82, Te-128 and Te-130, respectively, obtaining half-lives of around 10^{21} years for Te-130.

2.2. The period of grand unified theories

A renewed interest in double beta decay during the period of grand unified theories originates in its ability to test the symmetry properties of those theories such as LN conservation, etc. In many modern gauge theories beyond the standard model (SM) of particle physics, however, none of these symmetries are exact and they are violated to some degree depending on the model [14]. Double beta decay is expected to yield information on the degree of violation and to set important constraints on those models.

The requirement that both LN conservation and the invariance of left handedness of the weak current had to be violated in order for the $0\nu\beta\beta$ decay to occur, discouraged experimental searches for a period of time [8]. With the development of modern gauge theories during the last quarter of the previous century, perceptions began to change. In the SM of particle physics, which unifies the weak and electromagnetic interactions, it became apparent that the assumption of LN conservation led to the neutrino being strictly massless, thus preserving the left handedness of the weak current [4]. With the development of grand unified theories (GUTs) of the electroweak and strong interactions, the prejudice has grown that LN conservation was the result of a global symmetry, not a gauge symmetry, and had to be broken at some level [4]. In other words, modern GUTs and supersymmetric (SUSY) extensions of the SM suppose that such conservation laws of the SM may be violated to some small degree. The LN may only appear to be conserved at low energies because of the large grand unified mass scale governing its breaking. The considerations of the sensitivity of the $0\nu\beta\beta$ decay experiments to neutrino mass $\langle m_{\beta\beta}\rangle \sim 1$ eV became the genesis of a new interest in double beta decay.

Despite significant progress in the experimental techniques in 1960–70s, $2\nu\beta\beta$ decay was not observed until the 1987 in a laboratory setting by a group led by Michael Moe at the UC Irvine [15] for isotope Se-82. Since then many experiments have been conducted, which observed $2\nu\beta\beta$ decay in a number of other isotopes. $2\nu\beta\beta$ decay is the rarest known kind of radioactive decay; as of 2012, it has been observed for only 12 isotopes (including double electron capture in Ba-130 observed in 2001), and all of them have a mean lifetime of more than 10^{18} years [8]. However none of those experiments so far have confirmed positive results for the $0\nu\beta\beta$ decay process, pushing the lower bound for its half-life up to 10^{25} years, set by the Heidelberg–Moscow [16], GERDA [65, 66], EXO-200 [17–19] and KamLAND-Zen [20] experiments.

2.3. The period of Majorana neutrinos

Neutrino masses could be explained by the existence of right handed neutrinos, which could be either Dirac or Majorana particles [4], meaning that they are identical with their antiparticles up to a phase. The existence of the neutrino masses led to the violation of LN conservation principle, which led us to go beyond the SM of particle physics. A possible candidate for such a theory is the left-right symmetric model of grand unification inaugurated by Salam, Pati, Mohapatra and Senjanovi´c [21–23]. The $0\nu\beta\beta$ decay, which involves the emission of two electrons and no neutrinos, has been found as a powerful tool to study the LN conservation and to know whether neutrinos are Dirac or Majorana particles, since Schechter and

Valle proved that, if the $0\nu\beta\beta$ decay takes place, regardless of the mechanism causing it, the neutrinos are Majorana particles with non-zero mass and the LN is violated by 2 units [24, 25].

Various early measurements of neutrinos produced in the sun, in the atmosphere, and by accelerators suggested that neutrinos might oscillate from one 'flavor' (electron, muon and tau) to another, expected as a consequence of non-zero neutrino mass. More recently, neutrino oscillation results due to SuperKamiokande [26–29], SNO [30–36], KamLand [37] and other experiments, have yielded compelling evidence that the three observed flavors of neutrinos are not mass eigenstates but rather linear combinations of those eigenstates (at least two of which have nonzero mass eigenvalues) [38]. The experimental results could prove that the electron-neutrino mixes significantly with the neutrino mass eigenstates. In that case, the effective neutrino mass, $\langle m_{\beta\beta} \rangle$ could be large enough that the $0\nu\beta\beta$ decay may well be observed in experiments currently under construction or development [38]. An observation would establish that neutrinos are Majorana particles, the LN is violated by 2 units and a measurement of the decay rate, $\Gamma^{0\nu}$ combined with neutrino oscillation data, would yield insight into all three neutrino mass eigenstates.

CHAPTER III

OUR CURRENT KNOWLEDGE ABOUT NEUTRINO MASSES AND MIXINGS

The question of neutrino masses and mixing is one of the most important issues of modern particle and nuclear physics. It has already been discussed in a number of reviews [39–43] and its relevance to the $0\nu\beta\beta$ decay process will also be briefly discussed in this report.

3.1. Dirac versus Majorana neutrinos

The fact that the neutrino has no electric charge endows it with certain properties not shared by the charged fermions of the SM of particle physics. One can write two kinds of Lorentz invariant mass terms [43] for the neutrino, Dirac and Majorana masses, whereas for the charged fermions, conservation of electric charge allows only Dirac-type mass terms. In the four component notation for describing fermions, commonly used for writing the Dirac equation for the electron, the Dirac mass in the Lagrangian has the form, $m_D\overline{\psi}_R(x)\psi_L(x)$, connecting fields of opposite handedness (chirality), whereas the Majorana mass is of the form, $m_M\chi^T(x)C\chi(x)$, connecting fields of the same chirality, where, $\psi(x)$ and $\chi(x)$ represent the Dirac and the Majorana field, respectively, m_D and m_M are the masses of the Dirac and Majorana neutrinos, respectively, T denotes the transpose and C is the charge conjugation matrix. In the first case, the fermion, described by the field, $\psi(x)$ is different from its antiparticle, whereas in the latter case it is its own antiparticle up to a phase, i.e., a Majorana fermion is a Dirac fermion that is self-conjugate. Thus a Majorana field, $\chi(x)$ satisfies the following condition, known as Majorana condition [40]:

$$\chi_c(x) \equiv C\overline{\chi}^T(x) = \xi\chi(x), \tag{3.1.1}$$

where ξ is the Majorana phase factor.

A Majorana neutrino implies a whole new class of experimental signatures, the most prominent among them being the process of $0\nu\beta\beta$ decay of heavy nuclei. Since $0\nu\beta\beta$ decay arises due to the presence of Majorana neutrino masses, a measurement of its rate, $\Gamma^{0\nu}$ can provide information about neutrino masses and mixing, assuming (i) one can satisfactorily eliminate other contributions to this process that may arise from other interactions in a full beyond-the-standard-model theory, (ii) one can precisely estimate the values of the nuclear matrix elements, $M^{0\nu}$ associated with the $0\nu\beta\beta$ decay in question.

The expressions for the Dirac and Majorana mass terms make it clear that a theory forbids Majorana masses for a fermion only if there is an additional global symmetry under which it has nonzero charge [43]. For charged fermions such as the electron and the muon, Majorana mass-terms are forbidden by the fact that they have nonzero electric charge and the theory have electromagnetic $U(1)$ invariance. Hence all charged fermions are Dirac fermions. On the other hand, a Lagrangian with both Majorana and Dirac masses could describe a pair of Majorana fermions, irrespective of how small the Majorana mass term is (although it may prove very difficult to address whether the fermion is of the Dirac or the Majorana type when the Majorana mass-term is significantly smaller than the Dirac mass term) [43]. Although it is possible that neutrinos are Dirac fermions, modern GUTs and supersymmetric (SUSY) extensions of the SM suppose that neutrinos are Majorana fermions.

3.2. Neutrino mixings and neutrino masses

3.2.1. An overview

The recent discovery of neutrino oscillations [26, 44–46] has given the first evidence of physics beyond the SM of particle physics and in particular indicates that the neutrinos are massive particles. The oscillations were able to show that the neutrinos are admixed, determined two of the mixing angles and set a stringent limit on the third (for a global analysis see, e.g. [47]). Furthermore, they determined one square mass difference and the absolute value of the other. Neutrino oscillations, however, cannot determine the following [4]:

• Whether the neutrinos are Majorana or Dirac particles. It is obviously important to proceed further and decide on this important issue. $0\nu\beta\beta$ decay can achieve this, even if there might be processes that dominate over the conventional intermediate neutrino mechanism of $0\nu\beta\beta$ decay. It has been known that whatever the LN violating process is, which gives rise to $0\nu\beta\beta$ decay, it can be used to generate a Majorana mass for the neutrino [24]. This mechanism, however, may not be the dominant mechanism for generating the neutrino mass [48].

• The absolute scale of the neutrino masses. Neutrino oscillation experiments can measure only mass squared differences. This task can be accomplished by astrophysical observations or via other experiments involving low-energy weak decays, like triton decay or electron capture, or the $0\nu\beta\beta$ decay. It seems that for a neutrino mass in the meV region, the best process for achieving this is the $0\nu\beta\beta$ decay process.

• The neutrino mass hierarchy. Neutrino oscillation experiments cannot at present decide which scenario is realized in nature, namely the degenerate, the normal hierarchy (NH) or the inverted hierarchy (IH). They may be able to distinguish between the two hierarchies in the future by observing the sign of mass splitting given by atmospheric oscillation experiment.

Hence the observation of $0\nu\beta\beta$ beta decay, in addition to confirming the Majorana nature of the neutrinos, would give information on the absolute neutrino mass scale, and potentially also on the neutrino mass hierarchy and the Majorana phases appearing in the unitary neutrino mixing matrix, known as PMNS matrix. The deep significance of the process stems from the so-called "black-box theorem" which establishes that observing $0\nu\beta\beta$ decay implies that at least one neutrino is a Majorana particle in complete generality, regardless of the mechanism causing the decay [24, 25].

3.2.2. Neutrino mixings

From neutrino oscillation experiments it is established that a neutrino changes its flavor (electron, muon or tau) while it propagates through space. Neutrinos interact as flavor eigenstates (ν_e, ν_μ and ν_τ) but they propagate as mass eigenstates (ν_1, ν_2 and ν_3). These three flavor eigenstates evolve in a complicated oscillatory manner, because they carry three different mass eigenstates that are playing off against each other. Thus the three observed flavors of neutrinos are linear combination of neutrino mass eigenstates. The mixing of the mass eigenstates can be described by a 3×3 unitary mixing matrix, known as the Pontecorvo–Maki–Nakagawa–Sakata neutrino mixing matrix or PMNS matrix, in honor of these pioneers. The neutrino mixing matrix, U_{PMNS} transforms states with well-defined masses into states with well-defined flavors:

$$\begin{bmatrix} \nu_e(x) \\ \nu_\mu(x) \\ \nu_\tau(x) \end{bmatrix} = U_{PMNS} \begin{bmatrix} \nu_1(x) \\ \nu_2(x) \\ \nu_3(x) \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{bmatrix} \begin{bmatrix} \nu_1(x) \\ \nu_2(x) \\ \nu_3(x) \end{bmatrix}, \tag{3.2.1}$$

where $\nu_e(x)$, $\nu_\mu(x)$ and $\nu_\tau(x)$ denotes the electron-neutrino, muon-neutrino and tau-neutrino fields, respectively and $\nu_1(x)$, $\nu_2(x)$ and $\nu_3(x)$ represents the fields for the neutrino mass eigenstates with mass m_1, m_2 and m_3, respectively.

In terms of three neutrino mixing angles, θ_{12}, θ_{23} and θ_{31}, and CP-violating phase, δ, the neutrino mixing matrix can be parameterized as [4, 38, 43]:

$$U_{PMNS} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix}, \quad (3.2.2)$$

with
$c_{ij} = cos\theta_{ij}, \qquad s_{ij} = sin\theta_{ij}, \qquad i, j = 1,2,3.$

For the case of Majorana neutrino, two additional CP-violating Majorana phases, α_1 and α_2 arises, due to the Majorana condition, since, a Majorana particle is identical to its antiparticle, up to a phase. Thus in the literature [4, 38, 43], for Majorana particles, the neutrino mixing matrix, U_{PMNS} in $(2.2.2)$ is redefined by multiplying the expression for U_{PMNS} by a diagonal phase matrix, P containing those Majorana phases:

$$P = diag\left(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\delta}\right). \quad (3.2.3)$$

Neutrino oscillation experiments have already provided measurements for the neutrino mass-squared differences, as well as the mixing angles. The value of the mixing angle, θ_{12} and θ_{13} and the lower bound of θ_{23} are known from the experiments. From KamLand collaboration [49], $tan^2\theta_{12} = 0.452^{+0.035}_{-0.033}$, from Super-Kamiokande collaboration [50], $sin^2 2\theta_{23} > 0.92$ (90% CL) are found. Form Daya Bay [51], RENO [52] and Double Chooz [53] collaborations, the value of $sin^2 2\theta_{13}$ are found to be: $0.092 \pm 0.016\ (stat) \pm 0.005\ (syst)$, $0.103 \pm 0.013\ (stat) \pm 0.011\ (syst)$ and $0.085 \pm 0.029\ (stat) \pm 0.042\ (syst)$, respectively.

3.2.3. Neutrino masses

Given the current precision of neutrino oscillation experiments and the fact that neutrino oscillations are only sensitive to mass-squared differences, we have the following information about neutrino mass eigenstates from solar and atmospheric neutrino oscillation experiments.

$\Delta m_{sol}^2 = \Delta m_{12}^2 = m_2^2 - m_1^2 = 7.65_{-0.20}^{+0.13} \times 10^{-5}\ eV^2$ [47] and $\Delta m_{atm}^2 = |\Delta m_{23}^2| = |m_3^2 - m_2^2| = (2.43 \pm 0.13) \times 10^{-3}\ eV^2$ [54]. Note that we do not know the absolute scale of the neutrino mass and the sign of $m_3^2 - m_2^2$. Based on the above information, the following possible arrangements of the neutrino masses are allowed [4, 38, 43]:

(i) Normal hierarchy (NH), i.e., $m_1 < m_2 \ll m_3$:

In this scenario, the solar neutrino oscillation involves the two lighter levels, i.e., the two masses with the smaller splitting given by, Δm_{sol}^2 are smaller than the third mass. Since, $m_2 \ll m_3$, the mass splitting due to atmospheric oscillation is, $\Delta m_{atm}^2 = m_3^2 - m_2^2$.

In NH, the mass of the lightest neutrino ($m_{lightest} = m_1$) is unconstrained. The heavier masses (m_2 and m_3) could be determined in terms of the mass of the lightest neutrino, if the absolute scale of the mass of the lightest neutrino is known:

$$m_2 = \left(\Delta m_{sol}^2 + m_{lightest}^2\right)^{1/2}, \qquad m_3 = \left(\Delta m_{atm}^2 + m_{lightest}^2\right)^{1/2}. \tag{3.2.4}$$

(ii) Inverted hierarchy (IH), i.e., $m_3 \ll m_1 < m_2$:

In this scenario, the solar neutrino oscillation takes place between the two heavier levels, i.e., the two masses with the smaller splitting given by, Δm_{sol}^2 are larger than the third mass. Since, $m_3 \ll m_2$, the mass splitting due to atmospheric oscillation is, $\Delta m_{atm}^2 = m_2^2 - m_3^2$.

In IH, we have no information about the mass of the lightest neutrino ($m_{lightest} = m_3$). The heavier masses (m_1 and m_2) could be determined in terms of the mass of the lightest neutrino, if the absolute scale of the mass of the lightest neutrino is known:

$$m_1 = \left(\Delta m_{atm}^2 - \Delta m_{sol}^2 + m_{lightest}^2\right)^{1/2}, \qquad m_2 = \left(\Delta m_{atm}^2 + m_{lightest}^2\right)^{1/2}. \tag{3.2.5}$$

Figure 2 shows the Normal and Inverted neutrino mass spectrum.

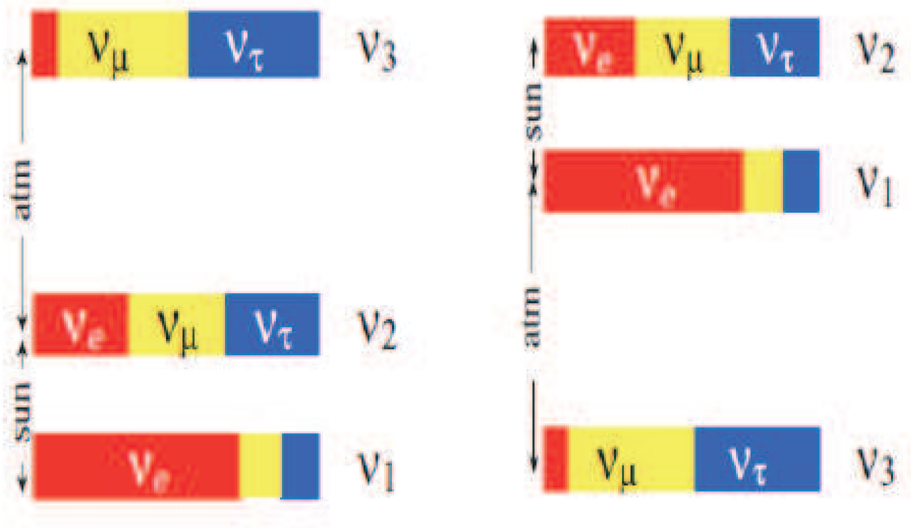

Figure 2. Normal and Inverted neutrino mass spectrum.
Taken from Mohapatra R. N. *et al*., *Rep. Prog. Phys.* **70**, 1757 (2007).

In addition, since the absolute mass scale is unknown, it is possible that the differences between the three mass eigenvalues are small compared to the masses themselves, that is, $m_1 \cong m_2 \cong m_3$. In this arrangement the neutrinos are referred to as quasidegenerate or sometimes simply as degenerate.

The mass of the lightest neutrino, $m_{lightest}$ and hence the absolute scale of the masses of all three neutrino mass eigenstates could be, in principle, extracted from $0\nu\beta\beta$ decay, from ordinary β decay and from astrophysical and cosmological observations, which are described briefly in the next three sections.

3.2.4. Neutrino masses from neutrinoless double beta decay

One of the probes for determining the absolute scale of the neutrino masses is the search for the $0\nu\beta\beta$ decay. For $0\nu\beta\beta$ decay process, the decay rate, $\Gamma^{0\nu}$ is proportional to the square of the effective Majorana neutrino mass, $\langle m_{\beta\beta} \rangle$, which is, in turn, related to the absolute mass scale and oscillation parameters through [40]:

$$\langle m_{\beta\beta} \rangle = \left| \sum_k (U_{ek}^L)^2 \, \xi_k m_k \right|, \tag{3.2.6}$$

where U_{ek}^L is the first row of the neutrino mixing matrix, U_{PMNS} and the ξ_k is the Majorana phase factor for the Majorana neutrino mass eigenstate with mass, m_k. We choose $\xi_k = \pm 1$.

The decay rate, $\Gamma^{0\nu}$ for the $0\nu\beta\beta$ decay process is potentially measurable if the neutrinos are Majorana fermions and $\langle m_{\beta\beta} \rangle$ is large enough [4, 38–40, 43], or if there are new LN violating interactions [43, 55–57]. In the absence of new LN violating interactions, a positive sign of $0\nu\beta\beta$ decay would allow one to measure $\langle m_{\beta\beta} \rangle$. Either way, we would learn that the neutrinos are Majorana fermions [24, 43, 58]. However, if $\langle m_{\beta\beta} \rangle$ is very small, and there are new LN violating interactions, $0\nu\beta\beta$ decay will measure the strength of the new interactions (such as doubly charged Higgs fields or R-parity violating interactions) rather than neutrino mass [43].

In the NH, the state corresponding to the largest mass contributes with a small mixing angle. Hence if the mass of the lightest neutrino, $m_{lightest}$ is small, $\langle m_{\beta\beta} \rangle$ is also small. By contrast, in the IH, the heavier neutrinos are the large contributors to $\langle m_{\beta\beta} \rangle$. If IH becomes the true scenario of neutrino mass spectrum, we will able to see it by the next generation $0\nu\beta\beta$ decay experiments [38]. On the other hand, any nonzero signal of $0\nu\beta\beta$ decay could restrict the parameter space available for the possibility of NH and if any sensitive experiment would detect very small $\langle m_{\beta\beta} \rangle$, it would demonstrate the possibility of NH [38]. However, one must be very careful in interpreting any nonzero signal in $0\nu\beta\beta$ decay experiments and not jump to the conclusion that a direct measurement of neutrino mass has been made because there are many examples of models of new LN violating interactions, which can lead to a $0\nu\beta\beta$ decay rate in the observable range without at the same time yielding a significant Majorana mass for the neutrinos [43]. The way to tell whether such a nonzero signal is due to neutrino masses or is a reflection of new interactions is, for example, to supplement $0\nu\beta\beta$ decay results with collider searches for these new interactions.

Figure 3 shows $\langle m_{\beta\beta}\rangle$ as a function of the lightest neutrino mass, $m_{lightest}$ both for the NH and IH mass spectra [4]. The figure shows, as expected, that $\langle m_{\beta\beta}\rangle$ is larger in the IH scenario than the NH scenario. Thus IH allowed region for $\langle m_{\beta\beta}\rangle$ is presented by the region between two parallel lines in the upper part of figure 3. The NH allowed region for $\langle m_{\beta\beta}\rangle \approx$ few meV is compatible with $m_{lightest}$ smaller than 10 meV. The plot shows the regions rather than the lines due to the unknown Majorana phases [38]. $0\nu\beta\beta$ decay is unable to distinguish the IH from the quasidegenerate spectrum as there is no way to measure $m_{lightest}$. The quasidegenerate spectrum can be determined if $m_{lightest}$ is known from the future β decay experiments such as KATRIN [59, 60] and MARE [61] or from cosmological observations.

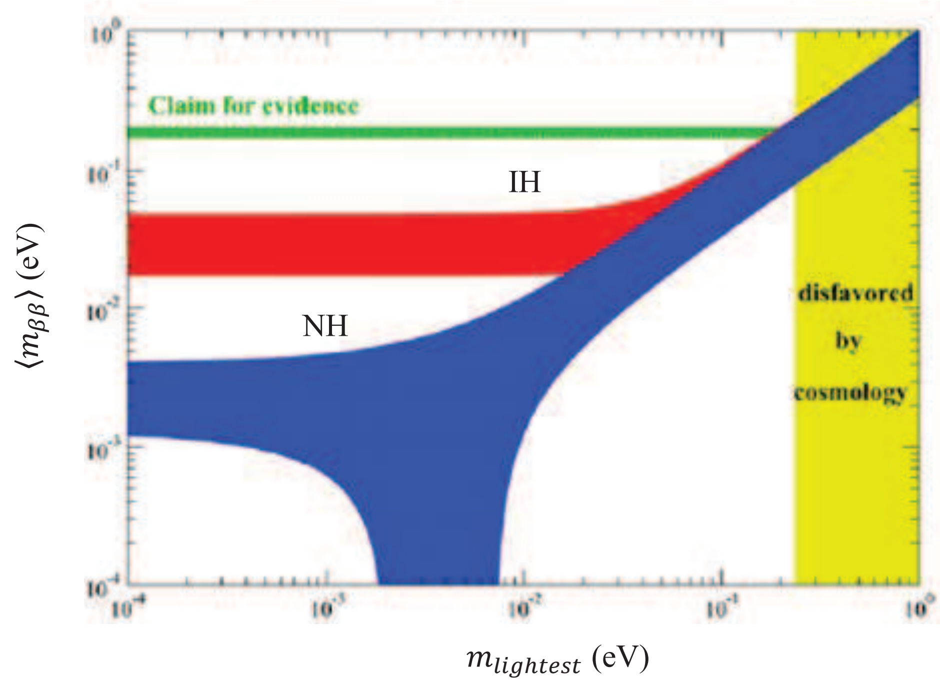

Figure 3. The allowed range of values of effective neutrino mass, $\langle m_{\beta\beta} \rangle$ for the NH and IH spectra as a function of the lightest neutrino mass, $m_{lightest}$ in eV.
Taken from Vergados J. D., *et al*., *Rep. Prog. Phys.* **75**, 106301 (2012). For NH spectrum, $m_{lightest} = m_1$ and for IH spectrum, $m_{lightest} = m_3$. Also shown are the claim of the discovery of the $0\nu\beta\beta$ decay set by the Heidelberg–Moscow collaboration [64] and the upper bound on the sum of neutrino masses [4, 38, 43] set by cosmology. Note that in the IH there is a lower bound, which means that in such a scenario the $0\nu\beta\beta$ decay should definitely be observed if the experiments reach the required level.

From the most precise experiments on the search for $0\nu\beta\beta$ decay [16, 19, 20] using the nuclear matrix elements (NMEs) of [63], the following upper bounds [4] of $\langle m_{\beta\beta} \rangle$ were inferred:

$\langle m_{\beta\beta} \rangle <$ (0.20–0.32) eV (Ge-76),

$\langle m_{\beta\beta} \rangle <$ (0.33–0.46) eV (Te-130) and

$\langle m_{\beta\beta} \rangle <$ (0.17–0.30) eV (Xe-136).

There is a claim of the discovery of the $0\nu\beta\beta$ decay of enriched Ge-76 experiment made by the Heidelberg–Moscow collaboration [64]. The experimental results from the new germanium enriched experiment GERDA Phase I [65] do not support the claim. Their estimated range [65] for the upper limit on value of the effective neutrino mass, $\langle m_{\beta\beta} \rangle$ is 0.2- 0.4 eV. In future experiments, CUORE [67], EXO [68, 69], MAJORANA [70], SuperNEMO [71], SNO+ [72], KamLAND-Zen and others [8, 38, 73], a sensitivity of $\langle m_{\beta\beta} \rangle \approx$ a few 10^{-2} eV is expected to be reached, which is the region of the IH of neutrino mass spectrum. In the case of the NH, $\langle m_{\beta\beta} \rangle$ is very small in order to be probed in the $0\nu\beta\beta$ decay experiments of the next generation.

3.2.5. Neutrino masses from ordinary beta decay

Another probe for the absolute scale of the neutrino masses regardless of whether the neutrinos are Dirac or Majorana particles is the direct search for the kinematic effect of nonzero neutrino masses in β decay by modifications of the Kurie plot [4, 38, 43]. Unlike the $0\nu\beta\beta$ decay, the decay rate, $\Gamma^{0\nu}$ of which reflects the coherent exchange of virtual Majorana neutrinos, β decay involves the emission of real neutrino, whose mass can alter the β particle spectrum. The corresponding effective beta decay mass, $\langle m_{\beta} \rangle$ reflects the incoherent sum of the masses of the neutrino mass eigenstates regardless of whether the neutrinos are Dirac or Majorana particles [4, 38, 43]:

$$\langle m_{\beta} \rangle = \sqrt{\sum_{k} |U_{ek}^{L}|^2 \, m_k^2}. \tag{3.2.7}$$

If the three neutrino masses are small, then the effects will occur near to the end point of the electron energy spectrum of the β decay process and will be sensitive to the effective β decay mass, $\langle m_{\beta} \rangle$ [43]. The quantity, $\langle m_{\beta} \rangle$ would approximate the difference between the end point of the electron energy spectrum and the Q value. The approximation is valid as long as the energy resolution is too poor to separate individual end points due to each of the m_k [38].

The Mainz [74] and Troitsk [75] experiments place the present upper limit on $\langle m_\beta \rangle \leq 2.3$ eV and 2.2 eV, respectively. The proposed KATRIN [59, 60] experiment is projected to be sensitive to $\langle m_\beta \rangle > 0.2$ eV, which will have important implications for the theory of neutrino masses. For instance, if the result is positive, it will imply a degenerate spectrum; on the other hand a negative result will be a very useful constraint [43]. In figure 4, $\langle m_\beta \rangle$ as a function of the lightest neutrino mass, $m_{lightest}$ is shown both for the NH and IH spectra of the β decay of tritium: ${}^3_1H \rightarrow {}^3_2He + e^- + \overline{\nu}_e$ [4]. Note that the dependence of $\langle m_\beta \rangle$ on the neutrino mass eigenvalues and mixing parameters, given by (3.2.7) is quite different than that of $\langle m_{\beta\beta} \rangle$, given by (3.2.6). The plot shows the lines rather than the regions due to the absence of unknown Majorana phases.

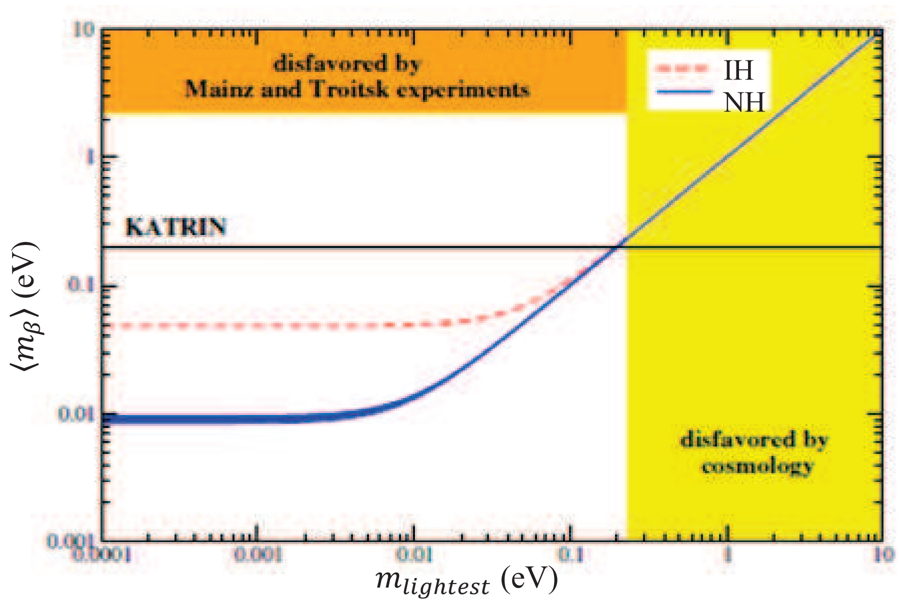

Figure 4. The effective neutrino mass, $\langle m_\beta \rangle$ in eV for the β decay of tritium both for the NH and IH spectra as a function of the lightest neutrino mass, $m_{lightest}$ in eV.
Taken from Vergados J. D., *et al.*, *Rep. Prog. Phys.* **75**, 106301 (2012). The current experimental upper limits on $\langle m_\beta \rangle$ from the Mainz [74] and Troitsk [75] experiments are 2.3 eV and 2.2 eV, respectively both for the NH and IH mass spectra. The sensitivity of the proposed KATRIN experiment is also shown in the figure. It is assumed that $\Delta m^2_{sol} = 7.65^{+0.13}_{-0.20} \times 10^{-5}\ eV^2$ [47],

$\Delta m^2_{atm} = (2.43 \pm 0.13) \times 10^{-3}\ eV^2$ [54], $tan^2\theta_{12} = 0.452^{+0.035}_{-0.033}$ [49] and $sin^2 2\theta_{13} = 0.092 \pm 0.016\ (stat) \pm 0.005\ (syst)$ [51].

3.2.6. Neutrino masses from Cosmology

The study of the cosmic microwave background (CMB) radiation spectrum as well as the study of the large scale structure in the universe also provides information on the absolute scale of neutrino masses. Astrophysical and cosmological observations set upper bounds on the sum of neutrino masses [4, 38, 43]:

$$\sum_k m_k = m_1 + m_2 + m_3 \leq m_{astro}. \quad (3.2.8)$$

The upper bound, m_{astro} depends on the type of observations made [76]. Hannestad [77] has emphasized that these upper limits can also change if there are more neutrino species. Surveys of large scale structure yields $m_{astro} = 0.7 - 2$ eV [77, 78]. Sloan Digital Sky Survey (SDSS) place the limit, $m_{astro} = 1.6$ eV [43]. CMB primordial gives 1.3 eV, CMB + distance gives 0.58 eV, galaxy distribution and lensing of galaxies gives 0.6 eV and the largest photometric redshift survey yields 0.28 eV [4]. Note that all of the above results are valid for both Majorana and Dirac neutrinos [43].

These limits for m_{astro} provide nontrivial information about neutrino masses. For instance, if the world average value of $m_{astro} = 0.71$ eV is taken at face value, this implies that each individual neutrino mass is smaller than 0.23 eV [43], which is similar to the projected sensitivity of the proposed KATRIN experiment. In figure 5, the sum of neutrino masses, $\sum_k m_k$ as a function of the lightest neutrino mass, $m_{lightest}$ is shown both for the NH and IH spectra, extracted from cosmology [4].

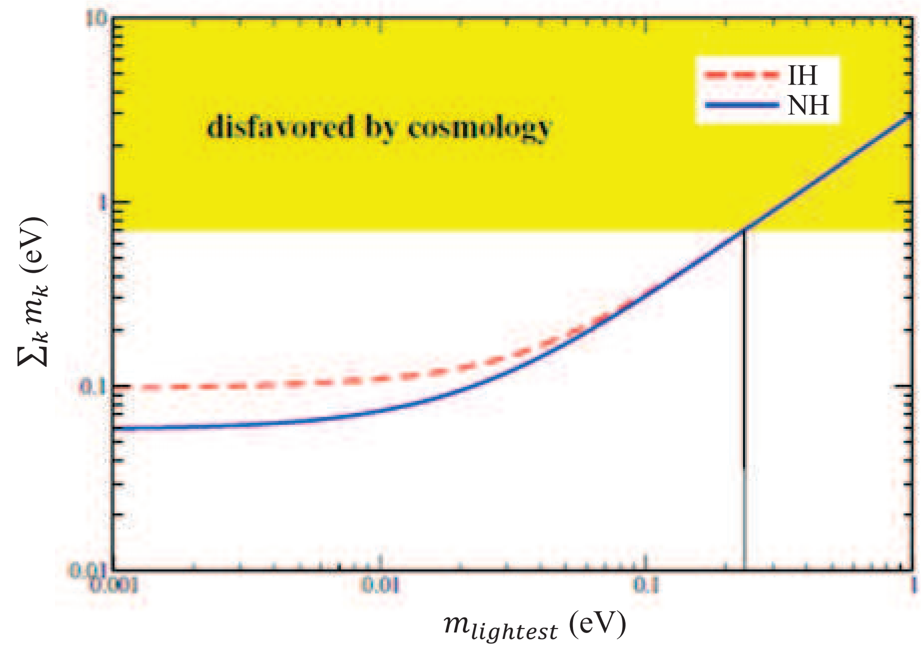

Figure 5. The sum of neutrino masses, $\sum_k m_k$ in eV, extracted from cosmology, both for the NH and IH spectra as a function of the lightest neutrino mass, $m_{lightest}$ in eV.
Taken from Vergados J. D., *et al*., *Rep. Prog. Phys.* **75**, 106301 (2012). From the world average value of astrophysical limit , $m_{astro} = 0.71$ eV, the corresponding neutrino mass extracted is about 0.23 eV, both for the NH and IH mass spectra. It is assumed that $\Delta m^2_{sol} = 7.65^{+0.13}_{-0.20} \times 10^{-5}\ eV^2$ [47], $\Delta m^2_{atm} = (2.43 \pm 0.13) \times 10^{-3}\ eV^2$ [54], $tan^2\theta_{12} = 0.452^{+0.035}_{-0.033}$ [49] and $sin^2 2\theta_{13} = 0.092 \pm 0.016\ (stat) \pm 0.005\ (syst)$ [51].

CHAPTER IV

QUANTUM FIELD THEORETICAL BACKGROUND

4.1. The interaction picture and the equations of motion

As we know from quantum mechanics, the theory can be formulated in several different but completely equivalent pictures or representations that differ in the way the time dependence is treated. Two extreme choices that can be constructed in this way are the Schrodinger picture and the Heisenberg picture. In Schrodinger picture, operators are constant but state vectors are time dependent. On the other hand, in Heisenberg picture, state vectors are constant but operators are time dependent.

It turns out that a third picture, the interaction picture is easier to use for interactions in quantum field theory [79, 80, 81]. The reason for that is: there is no way to find the exact solutions for interacting quantum field theories. Nevertheless, if the strength of the interaction is small, it is possible to obtain reliable approximate solutions using perturbation theory within the framework of interaction picture in place of trying to solve nonlinear, partial differential equations [79, 80, 81]. Moreover, the interaction picture allows us to analyze interacting fields using all the results developed in free Quantum field theory. We get this benefit by breaking the Hamiltonian into two parts.

As a starting point for the development of a perturbation theory, we assume that Hamiltonian, H of the system under study can be split into two parts:

$$H = H_0 + H_I. \tag{4.1.1}$$

Here, H_0 is typically the Hamiltonian for the free fields and H_I represents the interacting part of the Hamiltonian, H. For the free Hamiltonian, H_0, alone, an exact solution can be given. The strategy now is to put this 'easy' part of the problem into the definition of state vectors. Then only the perturbed Hamiltonian, H_I, will be visible in the equation of motion. This is achieved by a unitary transformation from the Schrodinger picture to the interaction picture so that both pictures lead to the same expectation values of the field operators [79, 80, 81]:

$$O^I(t) = e^{iH_0^S t} O^S e^{-iH_0^S t}, \qquad |\alpha, t\rangle^I = e^{iH_0^S t} |\alpha, t\rangle^S. \tag{4.1.2}$$

Similarly, the connection between the interaction picture and the Heisenberg picture is mediated by the following unitary transformations:

$$O^I(t) = e^{iH_0^S t} e^{-iHt} O^H e^{iHt} e^{-iH_0^S t}, \qquad |\alpha, t\rangle^I = e^{iH_0^S t} e^{-iHt} |\alpha, t\rangle^H, \tag{4.1.3}$$

where the superscripts $'S'$, $'H'$ and $'I'$ represent the Schrodinger, Heisenberg and the interaction pictures respectively. The subscripts $'o'$ and $'I'$ represent the free and interacting part of the Hamiltonian respectively. O and $|\alpha, t\rangle$ represent the operator and the state vector at time t respectively. In the interaction picture, the equation of motion for operators, O^I is given by:

$$i\frac{dO^I(t)}{dt} = [O^I(t), H_0^S], \tag{4.1.4}$$

where the bracket $'[,]'$ represents the commutator between two operators and the equation of motion for the state vectors, $|\alpha, t\rangle^I$ is given by:

$$i\frac{d}{dt}|\alpha, t\rangle^I = H_I^I |\alpha, t\rangle^I. \tag{4.1.5}$$

Thus in the interaction picture, we only need to solve only equation of motion $(4.1.5)$ for states which depends only on the interaction part, H_I^I of the Hamiltonian, H. The equation of motion for operators in the interaction picture given by $(4.1.4)$ depends only on the free part of the Hamiltonian. Importantly, the operator equations of motion in the interaction picture are the same as the equations of motion for operators in the Heisenberg picture for the free fields. This means that if we are working in the Heisenberg picture for the special case of free fields, where, $H = H_0$, then that special case equation of motion for an operator would be the same as the general case equation of motion for the operator in the interaction picture [79, 80, 81]. The free fields case like Klein-Gordon, Dirac, Maxwell equations of motion in the Heisenberg picture are the same as those in the interacting case in the interaction picture.

4.2. The S operator and the S matrix

The scattering matrix (S matrix) is a central concept in quantum field theory as well as in the ordinary quantum mechanics. In quantum field theory, we have many different interactions that can occur. Many different incoming particles can interact with one another and for every set of incoming particles, there are multiple final sets of outgoing particles.

These different interactions have different possibilities to occur. To keep track of each and every possible interactions, or more particularly, to keep track of the individual probabilities for each interaction to occur, we need something, called the S matrix, shorthand for scattering matrix, since each of these interactions can be considered as a scattering process. The square of the absolute value of each component of the S matrix, connecting a given initial and final state, equals the probability that transition taking place. For example:

$|S_{21}|^2 = probability\ of\ the\ 1\ st\ eigenstate\ transitioning\ to\ the\ 2\ nd.$

Thus, each component of the S matrix, is a transition amplitude for a particular reaction (scattering event, transition, interaction) between particles in particular eigenstates. For any interaction, we seek an operator, S, that is sandwiched between the initial state, $|i\rangle$ and the final state, $|f\rangle$, which gives us the transition amplitude, S_{fi} , from an initial state i to a final state f, that is, for the operator, S :

$$S_{fi} = \langle f|S|i\rangle. \tag{4.2.1}$$

4.3. The S operator for the state equation of motion in the interaction picture

There is a formal theory [79, 80, 81] which gives us the S operator within the framework of the interaction picture, discussed in the section 3.1. In the interaction picture, the equation of motion for the generic state vectors, $|\psi(t)\rangle^I$ is given by:

$$i\frac{d}{dt}|\psi(t)\rangle^I = H_I^I(t)|\psi(t)\rangle^I, \tag{4.3.1}$$

where, $H_I^I(t)$ represents the interacting part of the Hamiltonian in the interaction picture.

Since the initial state, $|i\rangle$ and the final state, $|f\rangle$ are related by the S operator in the following sense:

$$|f\rangle = \left|\psi(t_f)\right\rangle^I = S(t_f, t_i)|\psi(t_i)\rangle^I = S(t_f, t_i)|i\rangle, \tag{4.3.2}$$

it turns out that, in the interaction picture, the equation of motion for the operator, $S(t, t_i)$ is given by:

$$i\frac{d}{dt}S(t, t_i) = H_I^I(t)S(t, t_i). \tag{4.3.3}$$

The formal solution of (4.3.3) can be written as:

$$S(t_f, t_i) = exp\left(-i\int_{t_i}^{t_f} dt H_I^I(t)\right) = exp\left(-i\int_{t_i}^{t_f} d^4x \mathcal{H}_I^I(x)\right), \tag{4.3.4}$$

where, $\mathcal{H}_I^I(x)$ represents the Hamiltonian density corresponding to the interaction Hamiltonian, $H_I^I(t)$. The incorporation of the Hamiltonian density, $\mathcal{H}_I^I(x)$ comes through the fact that in the local field theory the Hamiltonian can be expressed as an integral over the Hamiltonian density.

For the case of particle decay, as time goes on, the probability of a decay being measured, increases. There are generally, a number of different possible final states, with different probabilities of being measured, at any given time. Consider the case for which, $t_i \to -\infty$ and $t_f \to \infty$. In this case, we would have a sort of equilibrium final states, in the sense that, the S operator would have had enough time to operate such that the probability of each possible final eigenstates would be fixed, and no longer changing with time. We would then have a certain probability for each particular final state to be measured.

Similarly, for three dimensional space, considering the integration time for the operation of the S operator to be very large, effectively infinite, will be advantageous in developing the formal theory of the S operator. Fortunately, the results using this assumption can be used to accurately predict real world experimental outcomes [79, 80, 81]. Thus the symbol, S represents an infinite space-time operator in the following sense:

$$S = \lim_{Volume \to \infty} S\left(t_f \to \infty, t_i \to -\infty\right) = exp\left(-i \int_{-\infty}^{\infty} d^4x \mathcal{H}_I^I(x)\right). \tag{4.3.5}$$

Following the idea of Dyson, the S operator, given in (4.3.5), can be expanded as a series, known as the Dyson expansion of the S operator [79, 80, 81]:

$$S = 1 - i \int_{-\infty}^{\infty} d^4x_1\, \mathcal{H}_I^I(x_1) + \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^4x_1 d^4x_2\, T\left[\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)\right] + \ \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} d^4x_1 d^4x_2 \ldots d^4x_n\, T\left[\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2) \ldots \mathcal{H}_I^I(x_n)\right] = \sum_{n=0}^{\infty} S^{(n)}, \tag{4.3.6}$$

where, T, denotes the time-ordered product, defined by:

$$T[\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)] = \begin{cases} \mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2) & for \quad t_1 > t_2 \\ \mathcal{H}_I^I(x_2)\mathcal{H}_I^I(x_1) & for \quad t_2 > t_1 \end{cases}. \tag{4.3.7}$$

If the $\mathcal{H}_I^I(x)$, in $(4.3.5)$, were numeric function of time, it wouldn't really matter what order, with respect to time, we carry out the integrations. However, since they are comprised of operators that act on a state vector to their right, we have to be sure that, at each point in the integration, the time wise earliest operators are acting first. The operation of the operators must occur in the order they occur in nature, in time sequence. So the earliest operator must operate on the state vector first, i.e., it must be furthest to the right. The next earliest to operate must be second from the right, etc. We indicate this ordering with the time ordering operator, T, defined by $(4.3.7)$.

CHAPTER V

WEAK INTERACTION AND THE DECAY RATE OF NEUTRINOLESS DOUBLE BETA DECAY PROCESS

5.1. Weak interaction and neutrinoless double beta decay

5.1.1. Weak-interaction Lagrangian

The SM of the electroweak interactions, due to Weinberg, Salam and Glashow, is perhaps the most illustrative example of a theory based on the use of concepts belonging to gauge theories and symmetry breaking mechanisms with a solid bearing with nature. In spite of its success the theory, however, relies upon a number of premises which may (or may not) remain valid if new scenarios about neutrino mass hierarchies, right handed interactions and lepton-flavor mixing are introduced. SM allows for quark-flavor mixing but not lepton-flavor mixing. It allows for left handed doublets and right handed singlet of leptons. Thus the SM can be considered as a low energy limit of a more general theory, which allows for lepton-flavor mixing and right handed doublets of leptons. Then the question about the experimental evidence of this more general theory immediately arises [82]. Nuclear double beta decay transitions are one of these experimental probes. However, the difficulties inherent in the detection of these decays are amplified by the difficulties related to the model dependent theoretical analysis of the data [82].

In the SM of electroweak interactions, the neutrinos are described as massless fermions, members of a left handed doublet. The absence of right handed neutrinos reflects upon the exclusion of LN violating electroweak transitions, like the $0\nu\beta\beta$ decay. In this fashion, only the $2\nu\beta\beta$ decay is allowed, since this decay mode is independent of the properties of the neutrinos [82]. The expression for the electroweak Lagrangian, in the $SU(2) \times U(1)$ representation, would thus acquire the familiar vector-axial vector ($V - A$) structure [83, 84]. A review of the possibilities offered by the current versions of the SM and its extensions can be found in Refs. [85, 86].

5.1.2. Weak beta decay Hamiltonian for neutrinoless double beta decay process

Since, it is well known that all existing data of the physics of weak and electromagnetic interactions are in wonderful agreement with the standard theory of electroweak interaction, in the discussions relevant to the nonzero neutrino masses, neutrino oscillations, etc., we shall assume, consequently, that the interaction of neutrinos with quarks and leptons is described by the Lagrangian of the standard theory, that is, that the "phenomenological" neutrinos and antineutrinos (in our case of $0\nu\beta\beta$ decay neutrinos and antineutrinos are identical, i.e., neutrinos are Majorana fermions) are particles which take part in the standard weak interaction.

The use of this Lagrangian and the associated weak interaction Hamiltonian density, $\mathcal{H}_w^\beta(x)$ in the context of the β decay has been explored in detail starting from the sixties [87, 88, 89] and the consequences of the $V - A$ structure of it, as determined by the observables of the nuclear single β decay, have been discussed in almost every text-books of particle physics. The action of $\mathcal{H}_w^\beta(x)$ on a pair of nucleons, to produce the sequential decay of them, can be described, say, in the $\beta^-\beta^-$ decay scenario, where from an initial nuclear state (the initial nuclear ground state) one nucleon decays into another nucleon and a pair of leptons (an electron, e^- and electron-neutrino, ν_e pair) is produced and the daughter nucleus is left in the ground state or an excited state (the states of the intermediate nucleus). This transition is a virtual one, since the ground state of the initial nucleus has a lower energy than the intermediate state consisting of the two leptons and the intermediate nucleus. The nuclear double beta decay takes place when a second virtual weak decay is considered between the intermediate and final nuclei. A second pair of leptons is produced and the final decay sequence can ($2\nu\beta\beta$ decay mode) or cannot ($0\nu\beta\beta$ decay mode) conserve LN. In the case of $0\nu\beta\beta$ decay, we finally ended up with two electrons and no neutrinos and a second order perturbative treatment of the standard weak β decay Hamiltonian is necessary to calculate the decay rate, $\Gamma^{0\nu}$. Figure 6 shows Feynman diagrams for for $2\nu\beta\beta$ and $0\nu\beta\beta$ decays.

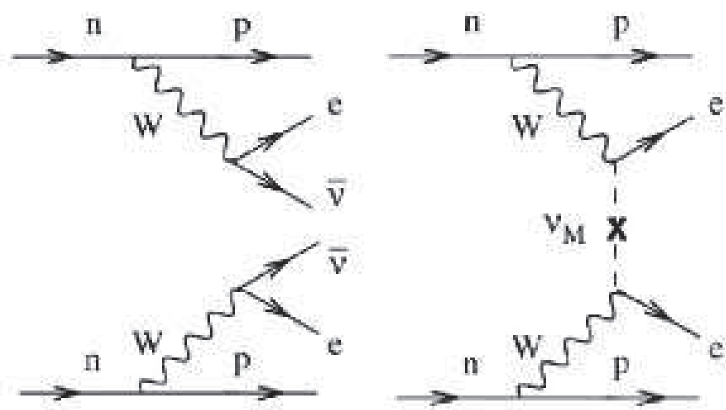

Figure 6. Feynman diagrams for $2\nu\beta\beta$ (left) and $0\nu\beta\beta$ (right) decays.
Taken from Avignone F. T., Elliott S. R., and Engel J., *Rev. Mod. Phys* . **80**, 481 (2008).

Within the framework of interaction picture, $0\nu\beta\beta$ decay is mediated by the effective weak β decay Hamiltonian density, $\mathcal{H}_w^{\beta}(x)$ having the standard form [40, 90]:

$$\mathcal{H}_w^{\beta}(x) = \frac{G_F}{\sqrt{2}} 2\overline{e}_L(x)\gamma^{\mu}\nu_{eL}(x) j_{\mu L}^{\dagger}(x) + H.c, \tag{5.1.1}$$

where G_F is the Fermi constant, $2\overline{e}_L(x)\gamma^{\mu}\nu_{eL}(x)$ is the leptonic current with $e(x)$ and $\nu_{eL}(x)$ representing field operators for electron and left-handed electron-neutrino, respectively and $j_{\mu L}^{\dagger}(x)$ is the strangeness conserving charged hadronic (nuclear) current in the interaction picture. $H.c$ denotes Hermitian conjugation.

Due to neutrino mixing, the electron-neutrino field, $\nu_{eL}(x)$ can be written as a linear combination of the fields, $\chi_{kL}(x)$ of the Majorana neutrino mass eigenstates, with mass, m_k:

$$\nu_{eL}(x) = \sum_k U_{ek}^L \chi_{kL}(x) = \sum_k U_{ek}^L \left(\frac{1-\gamma^5}{2}\right) \chi_k(x), \tag{5.1.2}$$

where U_{ek}^L is the first row of the unitary neutrino mixing matrix, U_{PMNS}, given by, (3.2.2). The fields, $\chi_k(x)$ satisfy the Majorana condition, given by:

$$C\bar{\chi}_k^T(x) = \xi_k \chi_k(x), \qquad \xi_k = \pm 1, \tag{5.1.3}$$

where, ξ_k's are the Majorana phase factors, C is the charge conjugation matrix and T denotes the transpose.

Using the definition of left-handed chiral spinor given in $(B.8)$ and $(B.9)$ in appendix B, and the anti-commutation property of the fifth gamma matrix, γ^5, given in $(A.2.7)$, the leptonic current, $2\bar{e}_L(x)\gamma^\mu \nu_{eL}(x)$ can be written in the standard vector-axial vector ($V - A$) form:

$$2\bar{e}_L(x)\gamma^\mu \nu_{eL}(x) = 2\bar{e}(x)\left(\frac{1+\gamma^5}{2}\right)\gamma^\mu \nu_{eL}(x) = \bar{e}(x)\gamma^\mu(1-\gamma^5)\nu_{eL}(x). \tag{5.1.4}$$

Thus, the leptonic current, $2\bar{e}_L(x)\gamma^\mu \nu_{eL}(x)$ consists of the electron field, $e(x)$, coupled to the left handed electron-neutrino field, $\nu_{eL}(x)$.

5.2. The S operator and the S matrix from the weak beta decay Hamiltonian

$0\nu\beta\beta$ decay occurs in second order of perturbation theory of weak interaction. Thus, the relevant term describing the $0\nu\beta\beta$ decay process, in the Dyson expansion of the S operator, given by, (4.3.6), is the second order term, $S^{(2)}$:

$$S^{(2)} = \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^4x_1 d^4x_2 \, T[\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)]. \tag{5.2.1}$$

Within the framework of the interaction picture, the Hamiltonian density, describing interaction, $\mathcal{H}_I^I(x)$, can be written as:

$$\mathcal{H}_I^I(x) = \mathcal{H}_w^\beta(x) + \mathcal{H}_s^I(x), \tag{5.2.2}$$

where $\mathcal{H}_w^\beta(x)$ is the weak β decay Hamiltonian density given by (5.1.1) and $\mathcal{H}_s^I(x)$ is the strong-interaction Hamiltonian density in the interaction picture. From (4.3.5) and (5.2.2), the S operator for the $0\nu\beta\beta$ decay can be written as:

$$S^{0\nu} = exp\left(-i\int_{-\infty}^{\infty} d^4x\mathcal{H}_w^\beta(x)\right) exp\left(-i\int_{-\infty}^{\infty} d^4x\mathcal{H}_s^I(x)\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} d^4x_1d^4x_2\dots d^4x_n\, T\left[\mathcal{H}_w^\beta(x_1)\mathcal{H}_w^\beta(x_2)\dots\mathcal{H}_w^\beta(x_n)\right] exp\left(-i\int_{-\infty}^{\infty} d^4x\mathcal{H}_s^I(x)\right), \tag{5.2.3}$$

and the corresponding second order term, $\left(S^{(2)}\right)^{0\nu}$, in the expansion of the $S^{0\nu}$ operator, is given by:

$$\left(S^{(2)}\right)^{0\nu} = \frac{(-i)^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^4x_1d^4x_2\, T\left[\mathcal{H}_w^\beta(x_1)\mathcal{H}_w^\beta(x_2)\right] exp\left(-i\int_{-\infty}^{\infty} d^4x\mathcal{H}_s^I(x)\right), \tag{5.2.4}$$

where T denotes the time-ordered product, defined in (4.3.7).

Since the time ordering on the leptonic current, $2\overline{e}_L(x)\gamma^\mu\nu_{eL}(x)$ and the hadronic current, $j_{\mu L}^\dagger(x)$ in the weak β decay Hamiltonian, $\mathcal{H}_w^\beta(x)$ act independently, the time-ordered product can be split into two parts:

$$T\left[\mathcal{H}_{weak}^\beta(x_1)\mathcal{H}_{weak}^\beta(x_2)\right] = 4\left(\frac{G_F}{\sqrt{2}}\right)^2 T\left[\overline{e}_L(x_1)\gamma^\mu\nu_{eL}(x_1)j_{\mu L}^\dagger(x_1)\overline{e}_L(x_2)\gamma^\mu\nu_{eL}(x_2)j_{\mu L}^\dagger(x_2)\right]$$

$$= 4\left(\frac{G_F}{\sqrt{2}}\right)^2 T\left[\overline{e}_L(x_1)\gamma^\mu\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^\nu)^T\overline{e}_L^T(x_2)\right]T\left[j_{\mu L}^\dagger(x_1)j_{\mu L}^\dagger(x_2)\right], \tag{5.2.5}$$

since,

$$\left(\overline{e}_L(x)\gamma^\mu\nu_{eL}(x)\right)^T = \nu_{eL}^T(x)(\gamma^\nu)^T\overline{e}_L^T(x).$$

By the application of Wick's theorem, (see appendix N for details),writing the time ordered product of the leptonic current in terms of the normal ordered product, the second order term, $\left(S^{(2)}\right)^{0\nu}$ is given by:

$$\left(S^{(2)}\right)^{0\nu} = -2\left(\frac{G_F}{\sqrt{2}}\right)^2 \int\int d^4x_1 d^4x_2 N\left[\overline{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T \overline{e}_L^T(x_2)\right]$$

$$\times T\left[j_{\mu L}^\dagger(x_1) j_{\nu L}^\dagger(x_2) exp\left(-i\int_{-\infty}^{\infty} d^4x \mathcal{H}_s^I(x)\right)\right], \qquad (5.2.6)$$

where N denotes the normal ordering of field operators, meaning all the annihilation operators should be placed to the right of all creation operators and $\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}$ is the Feynman propagator for the free electron-neutrino field, $\nu_{eL}(x)$.

Note that the Feynman propagator is only involved with the electron-neutrino fields, $\nu_{eL}(x)$. This is justified as electrons are created in the $0\nu\beta\beta$ decay process and neutrinos only exist in the intermediate stage of the decay process, i.e., they are created and absorbed.

5.3. Second-order perturbative matrix element for neutrinoless double beta decay process

In order to calculate the decay rate, $\Gamma^{0\nu}$ of the $0\nu\beta\beta$ decay process, we need to find an expression of the transition amplitude, $V_{fi}^{0\nu}$ for the nuclear transition from an initial state i to a final state f. Perturbation theory (see appendix O) allows us to extract $V_{fi}^{0\nu}$, from the second-order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$. This section is devoted to find an expression for $\langle f|S^{(2)}|i\rangle^{0\nu}$. Thus from (5.2.6), we can write down the matrix element corresponding to $\left(S^{(2)}\right)^{0\nu}$ term:

$$\langle f|S^{(2)}|i\rangle^{0\nu}$$

$$= -2\left(\frac{G_F}{\sqrt{2}}\right)^2 \int\int d^4x_1 d^4x_2 \left\langle p_2, s_2; p_1, s_1\middle| N\left[\bar{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T \bar{e}_L^T(x_2)\right]\middle|0\right\rangle$$

$$\times \langle p_f^I|T\left[j_{\mu L}^\dagger(x_1) j_{\nu L}^\dagger(x_2) exp\left(-i\int_{-\infty}^{\infty} d^4x \mathcal{H}_s^I(x)\right)\right]|p_i^I\rangle, \quad (5.3.1)$$

where $|0\rangle$ and $|p_2, s_2; p_1, s_1\rangle$ represents the vacuum state and the two-particle state for electrons with four-momentum, p_1, p_2 and spin projections, s_1, s_2 ,respectively. $|p_i^I\rangle$ and $|p_f^I\rangle$ represent the initial and final nuclear states in the interaction picture with four-momentum, p_i and p_f respectively.

Note that the strong interaction is taken into account in the interaction picture. For the hadronic current, the transition from the interaction picture to the Heisenberg picture is given by [91]:

$$\langle p_f^I|T\left[j_{\mu L}^\dagger(x_1) j_{\nu L}^\dagger(x_2) exp\left(-i\int_{-\infty}^{\infty} d^4x \mathcal{H}_{strong}^I(x)\right)\right]|p_i^I\rangle = \langle p_f|T\left[J_{\mu L}^\dagger(x_1) J_{\nu L}^\dagger(x_2)\right]|p_i\rangle, \quad (5.3.2)$$

where $|p_i\rangle$ and $|p_f\rangle$ represent the initial and final nuclear states in the Heisenberg picture with four-momentum, p_i and p_f respectively and $J_{\mu L}^\dagger(x)$ is the weak charged hadronic current in the Heisenberg picture.

Since for the free fields, equations of motion in the Heisenberg picture are the same as those in the interacting case in the interaction picture, leptonic matrix element in (5.3.1) remain unchanged under the transition from the interaction picture to the Heisenberg picture.

Assuming neutrinos are Majorana particles (5.1.3) and neutrino mixing does take place (5.1.2), the Feynman propagator $\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}$, for the free electron-neutrino fields, $\nu_{eL}(x)$ can be written as:

$$\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)} = -i\sum_k (U_{ek}^L)^2\, m_k\xi_k \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}\left(\frac{1-\gamma^5}{2}\right)C, \qquad (5.3.3)$$

where $q^\mu = (p-p')^\mu$ is the four-momentum transfer from hadrons to leptons, i.e., the four-momentum of the virtual neutrino.

Applying the quantization technique for the Dirac fields of electrons and substituting (5.3.2) and (5.3.3) into (5.3.1), the matrix element (see appendix F and G) can be written as:

$$\langle f|S^{(2)}|i\rangle^{0\nu}$$

$$= -i\left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U_{ek}^L)^2\, m_k\xi_k \overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)\int\int d^4x_1 d^4x_2\, e^{ip_1x_1+ip_2x_2}$$

$$\times\int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}\langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle - (p_1,s_1 \rightleftarrows p_2,s_2), \qquad (5.3.4)$$

where $u(p,s)$ represents the four- component free Dirac spinor for electrons with four-momenta, p and spin, s. $(p_1,s_1 \rightleftarrows p_2,s_2)$ denotes the fact that the second term is obtained by interchanging the four-momenta, p_1 and spin, s_1 with the four-momenta, p_2 and spin, s_2 of first term of (5.3.4).

The second term in (5.3.4) arises due to the identity of the final state electrons. It's not difficult to show (see appendix G) that the second term is identical to the first term. This can be done using the following identity,

$$\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2) = \left[\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)\right]^T$$

$$= -\overline{u}(p_2,s_2)\gamma^\nu(1-\gamma^5)\gamma^\mu C\overline{u}^T(p_1,s_1), \qquad (5.3.5)$$

as well as the possibility of interchanging the current operators under the sign of time ordered product.

Now, applying the residue theorem for contour integrals, the neutrino momentum integral in (5.3.4) can be evaluated as:

$$\int \frac{d^4q}{\pi(2\pi)^3}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2} = \begin{cases} -i\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{-iE_k(t_1-t_2)}e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k} & for \qquad t_1 > t_2 \\ -i\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{iE_k(t_1-t_2)}e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k} & for \qquad t_2 > t_1 \end{cases} . \qquad (5.3.6)$$

where $E_k = \sqrt{\vec{q}^2 + m_k^2}$ is the energy of the virtual neutrino.

To evaluate the time ordered product of the hadronic current, we split the hadronic current into space and time components, using the completeness relation for the nuclear states, $|n\rangle$:

$$\langle p_f|J^\dagger_{\mu L}(x_1)J^\dagger_{\nu L}(x_2)|p_i\rangle = \sum_n \langle p_f|J^\dagger_{\mu L}(\vec{x}_1)|n\rangle\langle n|J^\dagger_{\nu L}(\vec{x}_2)|p_i\rangle\, e^{i(E_f-E_n)t_1}e^{i(E_n-E_i)t_2}, \qquad (5.3.7)$$

where E_i, E_f and E_n are the energies of the initial, final and intermediate nuclear states, respectively.

Substituting (5.3.6) and (5.3.7) into (5.3.4), and performing the integration over the time variables using standard procedure of adiabatic switch-off of the interaction at $t \to \pm\infty$, we have the following expression for the matrix element of the $0\nu\beta\beta$ decay:

$$\langle f|S^{(2)}|i\rangle^{0\nu} = i\left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U^L_{ek})^2\, m_k \xi_k \overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)$$

$$\times \int\int d^3\vec{x}_1 d^3\vec{x}_2 e^{-i\vec{p}_1.\vec{x}_1 - i\vec{p}_2.\vec{x}_2}\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k}\times$$

$$\sum_n\left[\frac{\langle p_f|J^\dagger_{\mu L}(\vec{x}_1)|n\rangle\langle n|J^\dagger_{\nu L}(\vec{x}_2)|p_i\rangle}{E_n+E_k+E_2-E_i} + \frac{\langle p_f|J^\dagger_{\nu L}(\vec{x}_2)|n\rangle\langle n|J^\dagger_{\mu L}(\vec{x}_1)|p_i\rangle}{E_n+E_k+E_1-E_i}\right]2\pi\delta\left(E_f+E_1+E_2-E_i\right),$$

(5.3.8)

where $\vec{p}_1$ and $\vec{p}_2$ are the momenta and E_1 and E_2 are the energies of electrons, E_n is the energy associated with the intermediate nuclear state $|n\rangle$. Note in (5.3.8), the Dirac delta function represents the conservation of energy, i.e., $E_i = E_f + E_1 + E_2$.

The following two approximation will be made further for the second-order perturbative matrix element in (5.3.8):

(i) Closure approximation: Assuming the variations of energies of the intermediate nuclear states are negligible in compared to the energies of the emitting electrons and the energy of the virtual neutrino, the energies E_n of the intermediate states $|n\rangle$ are replaced by the average energy $E_n \to \langle E_n \rangle = E_j$. This allows us to perform the summation over the complete system of states $|n\rangle$ using the completeness relation for nuclear states.

(ii) Long wave approximations for leptonic waves: Since the de Broglie wavelengths associated with the momenta $\vec{p}_1$ and $\vec{p}_2$ of electrons are very small compared to the nuclear radius R_A of the decaying nuclei, i.e. $|\vec{p}_{1,2}|R_A \ll 1$, we can assume $e^{-i\vec{p}_1.\vec{x}_1 - i\vec{p}_2.\vec{x}_2} \to 1$.

With the above approximations, we finally have our second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$ for the $0\nu\beta\beta$ decay process:

$$\langle f|S^{(2)}|i\rangle^{0\nu} = i\left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U_{ek}^L)^2\, m_k \xi_k \overline{u}(p_1, s_1)(1+\gamma^5)C\overline{u}^T(p_2, s_2)$$

$$\times \left\langle \psi_f \middle| \int\int d^3\vec{x}_1 d^3\vec{x}_2 \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k}\left[\frac{1}{E_k + E_j + E_1 - E_i} + \frac{1}{E_k + E_j + E_2 - E_i}\right] J_{\mu L}^{\dagger}(\vec{x}_1) J_L^{\mu\dagger}(\vec{x}_2) \middle| \psi_i \right\rangle$$

$$\times\, 2\pi\delta(E_f + E_1 + E_2 - E_i). \tag{5.3.9}$$

where ψ_i and ψ_f are the wave functions of the initial and final nuclear states.

5.4. Calculation of the higher order terms in the nucleon current

The hadronic (nuclear) current, $J_L^\mu(x)$ can be obtained phenomenologically by imposing symmetry requirements to the more general combination that can be built with the Lorentz vectors. We also need to assume the impulse approximation, i.e., nucleons in nuclei can be treated as free when dealing with the weak interaction. Imposing Lorentz, parity and time-reversal invariance and assuming light-neutrino exchange throughout, within the impulse approximation, the hadronic current, $J_L^\mu(x)$, can be expressed in terms of nucleon fields, $\psi(x)$ [90]:

$$J_L^{\mu\dagger}(x) = \overline{\psi}(x)\tau^- \left[g_V(q^2)\gamma^\mu - g_M(q^2) i\frac{\sigma^{\mu\nu}}{2m_p} q_\nu - g_A(q^2)\gamma^\mu\gamma^5 + g_P(q^2) q^\mu \gamma^5\right]\psi(x) \quad (5.4.1)$$

where τ^- is the isospin lowering operator, i.e., it turns a neutron into a proton, m_P is the nucleon mass, $q^\mu = (p - p')^\mu$ is the four-momentum transfer from hadrons to leptons, i.e., the four-momentum of the virtual neutrino with p and p' being the four momenta of neutron and proton, respectively and, $\sigma^{\mu\nu}$ is the antisymmetric tensor, defined by:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \quad (5.4.2)$$

$g_V(q^2), g_M(q^2), g_A(q^2)$ and $g_P(q^2)$ are real functions of a Lorenz scalar, q^2. The values of these form factors in the zero-momentum transfer limit are known as the vector, weak-magnetism, axial-vector, and induced pseudoscalar coupling constants, respectively. Using dipole approximation, the form factors, $g_V(q^2)$ and $g_A(q^2)$ are given by [90, 92]:

$$g_V(q^2) = \frac{g_V}{(1 + q^2/\Lambda_V^2)^2}\,, \qquad g_A(q^2) = \frac{g_A}{(1 + q^2/\Lambda_A^2)^2}, \quad (5.4.3)$$

Here, $g_V = 1$ and $g_A = 1.254$ are the vector and axial constants. $\Lambda_V = 850$ MeV and $\Lambda_A = 1086$ MeV are the finite-size parameters. The weak-magnetism coupling constant, $g_M(q^2)$ is given by [90, 92]:

$$g_M(q^2) = (\mu_P - \mu_n) g_V(q^2), \tag{5.4.4}$$

where the magnetic moments, $(\mu_P - \mu_n) = 4.7$. The induced pseudoscalar coupling constant, $g_p(q^2)$ is given by the partially conserved axial-vector current hypothesis (PCAC) [90]:

$$g_P(q^2) = 2m_P \frac{g_A(q^2)}{q^2 + m_\pi^2}\left(1 - \frac{m_\pi^2}{\Lambda_A^2}\right). \tag{5.4.5}$$

Table 1. Non-relativistic expansion of two-component free spinor matrix elements in the nucleon current in the Breit frame.

Type	Bilinear combination of Dirac gamma matrices	$M(p',p)$
Pq^0	$g_P(q^2)q^0\gamma^5$	$g_P(q^2)\frac{q^0(\vec{\sigma}_n.\vec{q})}{2m_P} \approx 0$
$P\vec{q}$	$g_P(q^2)\vec{q}\gamma^5$	$g_P(q^2)\frac{\vec{q}(\vec{\sigma}_n.\vec{q})}{2m_P}$
V^0	$g_V(q^2)\gamma^0$	$\approx g_V(q^2)$
$\vec{V}$	$g_V(q^2)\vec{\gamma}$	$-g_V(q^2)i\frac{\vec{\sigma}_n \times \vec{q}}{2m_P}$
A^0	$g_A(q^2)\gamma^0\gamma^5$	0
$\vec{A}$	$g_A(q^2)\vec{\gamma}\gamma^5$	$\approx g_A(q^2)\vec{\sigma}_n$
$T^{0j}q_j$	$g_M(q^2)i\frac{\sigma^{0j}}{2m_p}q_j$	$-g_M(q^2)\frac{q^j q_j}{2m_P} = -g_M(q^2)\frac{\vec{q}^2}{2m_P} \approx 0$
$T^{ij}q_j$	$g_M(q^2)i\frac{\sigma^{ij}}{2m_p}q_j$	$\approx g_M(q^2)i\frac{\sigma_n^k \varepsilon_{ijk} q_j}{2m_P} = -g_M(q^2)i\frac{\varepsilon_{ijk} q^j \sigma_n^k}{2m_P}$ $= g_M(q^2)i\frac{[\vec{\sigma}_n \times \vec{q}]^i}{2m_P}$

It is necessary to reduce the nucleon current to the nonrelativistic form for nuclear structure calculations. Neglecting small energy transfers between nucleons in the nonrelativistic expansion, the two-component free spinor matrix elements in the nucleon current are given using the Breit frame [93] in the table 1.

Reducing the nucleon current from equation (5.4.1) to non-relativistic form in the Breit frame yields:

$$J_L^{\mu\dagger}(\vec{x}) = \sum_n \tau_n^- [g^{\mu 0} J^0(q^2) + g^{\mu k} J_n^k(\vec{q}^2)] \delta(\vec{x} - \vec{x}_n)\,, k = 1,2,3,$$

with

$$J^0(q^2) = g_V(q^2),$$

and

$$\vec{J}_n(\vec{q}^2) = g_M(q^2) i \frac{\vec{\sigma}_n \times \vec{q}}{2m_P} + g_A(q^2)\vec{\sigma}_n - g_P(q^2)\frac{\vec{q}(\vec{\sigma}_n.\vec{q})}{2m_P}. \qquad (5.4.6)$$

Here, $\vec{x}_n$ is the coordinate of the nth nucleon and $g^{\mu\nu} = diag(1,-1,-1,-1)$ is the metric tensor. From equation (5.4.6) we have:

$$J_{\mu L}^{\dagger}(\vec{x}_1) J_L^{\mu\dagger}(\vec{x}_2) = \sum_{n,m} \tau_n^- \tau_m^- \left[J^0(q^2)J^0(q^2) - \vec{J}_n(\vec{q}^2).\vec{J}_m(\vec{q}^2)\right] \delta(\vec{x}_1 - \vec{x}_n)\delta(\vec{x}_2 - \vec{x}_n),$$

with

$$J^0(q^2)J^0(q^2) = g_V^2(q^2),$$

and

$$\vec{J}_n(\vec{q}^2).\vec{J}_m(\vec{q}^2) = \left[g_M(q^2) i \frac{\vec{\sigma}_n \times \vec{q}}{2m_P} + g_A(q^2)\vec{\sigma}_n - g_P(q^2)\frac{\vec{q}(\vec{\sigma}_n.\vec{q})}{2m_P}\right].$$

$$\left[-g_M(q^2) i \frac{\vec{\sigma}_m \times \vec{q}}{2m_P} + g_A(q^2)\vec{\sigma}_m - g_P(q^2)\frac{\vec{q}(\vec{\sigma}_m.\vec{q})}{2m_P}\right]$$

$$= g_M^2(q^2)\frac{(\vec{\sigma}_n \times \vec{q}).(\vec{\sigma}_m \times \vec{q})}{4m_P^2} + g_A^2(q^2)\vec{\sigma}_n.\vec{\sigma}_m - g_A(q^2)g_P(q^2)\frac{(\vec{\sigma}_n.\vec{q})(\vec{\sigma}_m.\vec{q})}{m_P}$$

$$+ g_P^2(q^2)\frac{\vec{q}^2(\vec{\sigma}_n.\vec{q})(\vec{\sigma}_m.\vec{q})}{4m_P^2}. \qquad (5.4.7)$$

Using vector identity, the term, $(\vec{\sigma}_n \times \vec{q}).(\vec{\sigma}_m \times \vec{q})$ can be written as:

$$(\vec{\sigma}_n \times \vec{q}).(\vec{\sigma}_m \times \vec{q}) = \vec{q}^2(\vec{\sigma}_n.\vec{\sigma}_m) - (\vec{\sigma}_n.\vec{q})(\vec{\sigma}_m.\vec{q}) = \vec{q}^2(\vec{\sigma}_n.\vec{\sigma}_m) - \vec{q}^2(\vec{\sigma}_n.\hat{q})(\vec{\sigma}_m.\hat{q}), \quad (5.4.8)$$

where, $\hat{q}$ is a unit vector in the direction of $\vec{q}$. From equations (5.4.7) and (5.4.8), we have:

$$J^0(q^2)J^0(q^2) - \vec{J}_n(\vec{q}^2).\vec{J}_m(\vec{q}^2)$$

$$= g_V^2(q^2) - \left\{ g_A^2(q^2) - \frac{1}{3}\frac{g_A(q^2)g_P(q^2)}{m_P}\vec{q}^2 + \frac{1}{3}\frac{g_P^2(q^2)}{4m_P^2}\vec{q}^4 + \frac{2}{3}g_M^2(q^2)\frac{\vec{q}^2}{4m_P^2} \right\}\vec{\sigma}_n.\vec{\sigma}_m$$

$$+ \left\{ \frac{1}{3}\frac{g_A(q^2)g_P(q^2)}{m_P}\vec{q}^2 - \frac{1}{3}\frac{g_P^2(q^2)}{4m_P^2}\vec{q}^4 + \frac{1}{3}g_M^2(q^2)\frac{\vec{q}^2}{4m_P^2} \right\}[3(\vec{\sigma}_n.\hat{q})(\vec{\sigma}_m.\hat{q}) - \vec{\sigma}_n.\vec{\sigma}_m]. \quad (5.4.9)$$

Substituting the form factor, $g_P(q^2)$ from equations (5.4.5), into the equation(5.4.9), we get:

$$J^0(q^2)J^0(q^2) - \vec{J}_n(\vec{q}^2).\vec{J}_m(\vec{q}^2)$$

$$= g_V^2(q^2) - \left\{ g_A^2(q^2)\left[1 - \frac{2}{3}\frac{\vec{q}^2}{q^2 + m_\pi^2}\left(1 - \frac{m_\pi^2}{\Lambda_A^2}\right) + \frac{1}{3}\left(\frac{\vec{q}^2}{q^2 + m_\pi^2}\right)^2\left(1 - \frac{m_\pi^2}{\Lambda_A^2}\right)^2\right]\right.$$

$$\left. + \frac{2}{3}g_M^2(q^2)\frac{\vec{q}^2}{4m_P^2}\right\}\vec{\sigma}_n.\vec{\sigma}_m$$

$$+ \left\{ g_A^2(q^2)\left[\frac{2}{3}\frac{\vec{q}^2}{q^2 + m_\pi^2}\left(1 - \frac{m_\pi^2}{\Lambda_A^2}\right) - \frac{1}{3}\left(\frac{\vec{q}^2}{q^2 + m_\pi^2}\right)^2\left(1 - \frac{m_\pi^2}{\Lambda_A^2}\right)^2\right]\right.$$

$$\left. + \frac{1}{3}g_M^2(q^2)\frac{\vec{q}^2}{4m_P^2}\right\}[3(\vec{\sigma}_n.\hat{q})(\vec{\sigma}_m.\hat{q}) - \vec{\sigma}_n.\vec{\sigma}_m]. \quad (5.4.10)$$

Neglecting the factor, $m_\pi^2/\Lambda_A^2 \approx 0$ and using the operator, S_{nm}^q, defined by:

$$S_{nm}^q = 3(\vec{\sigma}_n.\hat{q})(\vec{\sigma}_m.\hat{q}) - \vec{\sigma}_n.\vec{\sigma}_m, \quad (5.4.11)$$

equation (5.4.10) can be written as:

$$J^0(q^2)J^0(q^2) - \vec{J}_n(\vec{q}^2).\vec{J}_m(\vec{q}^2) = g_A^2\left[\left(\frac{g_V}{g_A}\right)^2 h_F(q^2) - h_{GT}(q^2)\vec{\sigma}_n.\vec{\sigma}_m + h_T(q^2)S_{nm}^q\right]. \quad (5.4.12)$$

Thus substituting equation (5.4.12) into equation (5.4.7), we get:

$$J_{\mu L}^{\dagger}(\vec{x}_1)J_L^{\mu\dagger}(\vec{x}_2)$$

$$= -\sum_{n,m} \tau_n^- \tau_m^- g_A^2 \left[h_{GT}(q^2)\vec{\sigma}_n.\vec{\sigma}_m - \left(\frac{g_V}{g_A}\right)^2 h_F(q^2) - h_T(q^2) S_{nm}^q \right] \delta(\vec{x}_1 - \vec{x}_n)\delta(\vec{x}_2 - \vec{x}_n), \tag{5.4.13}$$

where the form factors, $h_{GT}(q^2), h_F(q^2)$ and $h_T(q^2)$ corresponding to the Gamow-Teller, Fermi and Tensor nuclear matrix elements, have the following form:

$$h_F(q^2) = \frac{g_V^2(q^2)}{g_V^2},$$

$$h_{GT}(q^2) = \frac{g_A^2(q^2)}{g_A^2}\left[1 - \frac{2}{3}\frac{\vec{q}^2}{q^2 + m_\pi^2} + \frac{1}{3}\left(\frac{\vec{q}^2}{q^2 + m_\pi^2}\right)^2\right] + \frac{2}{3}\frac{g_M^2(q^2)}{g_A^2}\frac{\vec{q}^2}{4m_P^2},$$

$$h_T(q^2) = \frac{g_A^2(q^2)}{g_A^2}\left[\frac{2}{3}\frac{\vec{q}^2}{q^2 + m_\pi^2} - \frac{1}{3}\left(\frac{\vec{q}^2}{q^2 + m_\pi^2}\right)^2\right] + \frac{1}{3}\frac{g_M^2(q^2)}{g_A^2}\frac{\vec{q}^2}{4m_P^2}. \tag{5.4.14}$$

The transition amplitude, $V_{fi}^{0\nu}$ is an essential ingredient to calculate the decay rate, $\Gamma^{0\nu}$. Assuming the closure and long wave approximations, the transition amplitude, $V_{fi}^{0\nu}$ for the $0_i^+ \rightarrow 0_f^+$, ground state to ground state transition can be obtained by comparing the expression for the second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$, given by (5.3.9), with the following expression coming from perturbation theory:

$$\langle f|S^{(2)}|i\rangle^{0\nu} = i2\pi\delta(E_f + E_1 + E_2 - E_i)V_{fi}^{0\nu}. \tag{5.4.15}$$

Hence the transition amplitude, $V_{fi}^{0\nu}$ for the $0\nu\beta\beta$ decay process, is given by:

$$V_{fi}^{0\nu} = \left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U_{ek}^L)^2 m_k \xi_k \overline{u}(p_1, s_1)(1+\gamma^5)C\overline{u}^T(p_2, s_2)$$

$$\times \left\langle \psi_f \middle| \int\int d^3\vec{x}_1 d^3\vec{x}_2 \int \frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{i\vec{q}.(\vec{x}_1 - \vec{x}_2)}}{E_k}\left[\frac{1}{E_k + E_j + E_1 - E_i} + \frac{1}{E_k + E_j + E_2 - E_i}\right] J_{\mu L}^{\dagger}(\vec{x}_1)J_L^{\mu\dagger}(\vec{x}_2) \middle| \psi_i \right\rangle. \tag{5.4.16}$$

where, G_F is the Fermi constant, U^L_{ek} is the first row of the neutrino mixing matrix, U_{PMNS} and ξ_k is the Majorana phase factor for the Majorana neutrino mass eigenstate with mass, m_k, $\vec{p}_1$, $\vec{p}_2$ and E_1, E_2 are the momenta and energies of electrons respectively, $\vec{q}$ and E_k are, respectively, the momenta and energy of Majorana neutrino mass eigenstate with mass, m_k, ψ_i and ψ_f are the wave functions of the initial and final nuclear state, E_i and E_j are the energies of the initial and intermediate nuclear states.

Assuming on an average, both electron shares the same energy, we have:

$$E_1 = E_2 = (E_i - E_f)/2, \qquad (5.4.17)$$

where E_f is the energy of the final nuclear state of the of the neutrinoless double beta decay. Typically, $|\vec{q}| = \mathrm{q} \approx 35$ MeV, on the other hand, $m_k \leq$ a few MeV, thus we can neglect the neutrino mass, m_k, i.e.,

$$E_k = \sqrt{\vec{q}^2 + m_k^2} \approx |\vec{q}| = \mathrm{q}, \qquad (5.4.18)$$

Substituting equation (5.4.17) and the approximation given by (5.4.18) into equation (5.4.16), we get:

$$V_{fi}^{0\nu} = \frac{m_e}{2\pi}\left(\frac{G_F}{\sqrt{2}}\right)^2 \frac{1}{R_A}\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)\frac{R_A}{2\pi^2} \times$$

$$\left\langle \psi_f \middle| \int\int\int d^3\vec{x}_1 d^3\vec{x}_2 d^3\vec{q}\, \frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{\mathrm{q}} \frac{1}{\mathrm{q} + E_j - (E_i + E_f)/2} J^{\dagger}_{\mu L}(\vec{x}_1) J^{\mu\dagger}_L(\vec{x}_2) \middle| \psi_i \right\rangle \left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right), \qquad (5.4.19)$$

where m_e is the mass of electron, $\langle m_{\beta\beta}\rangle$ is the effective neutrino mass given by (3.2.6), R_A is the radius of the initial nucleus of mass number, A.

Substituting $J^{\dagger}_{\mu L}(\vec{x}_1) J^{\mu\dagger}_L(\vec{x}_2)$ from equation (5.4.13) into equation (5.4.19) and integrating over $d^3\vec{x}_1$ and $d^3\vec{x}_2$, we have:

$$V_{fi}^{0\nu} = -\frac{g_A^2 m_e}{2\pi}\left(\frac{G_F}{\sqrt{2}}\right)^2 \frac{1}{R_A}\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)\frac{R_A}{2\pi^2}\times$$

$$\left\langle\psi_f\right|\textstyle\sum_{n,m}\int d^3\vec{q}\,\frac{e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}}{\mathrm{q}}\frac{1}{\mathrm{q}+E_j-(E_i+E_f)/2}\tau_n^-\tau_m^-\left[h_{GT}(q^2)\vec{\sigma}_n.\vec{\sigma}_m-\left(\frac{g_V}{g_A}\right)^2 h_F(q^2)-h_T(q^2)S_{nm}^q\right]\left|\psi_i\right\rangle\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)$$

$$= -\frac{g_A^2 m_e}{2\pi}\left(\frac{G_F}{\sqrt{2}}\right)^2 \frac{1}{R_A}\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)\left(M_{GT}^{0\nu}-\left(\frac{g_V}{g_A}\right)^2 M_F^{0\nu}+M_T^{0\nu}\right)\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right), \tag{5.4.20}$$

where,

$$M_{GT}^{0\nu} = \frac{R_A}{2\pi^2}\left\langle\psi_f\right|\textstyle\sum_{n,m}\int d^3\vec{q}\,\frac{e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}}{\mathrm{q}}\frac{1}{\mathrm{q}+E_j-(E_i+E_f)/2}\tau_n^-\tau_m^-\,h_{GT}(q^2)\vec{\sigma}_n.\vec{\sigma}_m\left|\psi_i\right\rangle,$$

$$M_{F}^{0\nu} = \frac{R_A}{2\pi^2}\left\langle\psi_f\right|\textstyle\sum_{n,m}\int d^3\vec{q}\,\frac{e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}}{\mathrm{q}}\frac{1}{\mathrm{q}+E_j-(E_i+E_f)/2}\tau_n^-\tau_m^-\,h_{F}(q^2)\left|\psi_i\right\rangle,$$

$$M_{T}^{0\nu} = -\frac{R_A}{2\pi^2}\left\langle\psi_f\right|\textstyle\sum_{n,m}\int d^3\vec{q}\,\frac{e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}}{\mathrm{q}}\frac{1}{\mathrm{q}+E_j-(E_i+E_f)/2}\tau_n^-\tau_m^-\,h_{T}(q^2)S_{nm}^q\left|\psi_i\right\rangle, \tag{5.4.21}$$

are the Gamow-Teller, Fermi and Tensor nuclear matrix elements respectively which can be written (See Appendix K for details) in the following form:

$M_\alpha^{0\nu} = \langle\psi_f|O_\alpha|\psi_i\rangle\,,\alpha = \{GT, F, T\}\,,$

with

$$O_{GT} = \sum_{n,m}\tau_n^-\tau_m^-\,(\vec{\sigma}_n.\vec{\sigma}_m)H_{GT}(X_{nm},E_j)\,, O_F = \sum_{n,m}\tau_n^-\tau_m^-\,H_F(X_{nm},E_j)\,,$$

$$O_T = \sum_{n,m}\tau_n^-\tau_m^-\,S_{nm}H_T(X_{nm},E_j)\,,$$

where

$S_{nm} = 3(\vec{\sigma}_n.\hat{x}_{nm})(\vec{\sigma}_m.\hat{x}_{nm}) - (\vec{\sigma}_n.\vec{\sigma}_m)\,, \vec{x}_{nm} = \vec{x}_n - \vec{x}_m\,, x_{nm} = |\vec{x}_{nm}|\,\,, \hat{x}_{nm} = \vec{x}_{nm}/x_{nm}$

and

$$H_\alpha(r,E_j)=\frac{2R_A}{\pi}\int_0^\infty \frac{f_\alpha(\mathrm{q}r)h_\alpha(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q}+E_j-(E_i+E_f)/2}\ , \qquad (5.4.22)$$

where, $f_{GT,F}(\mathrm{q}r)=j_0(\mathrm{q}r)$ and $f_T(\mathrm{q}r)=j_2(\mathrm{q}r)$ are spherical Bessel functions and $h_\alpha(q^2)$'s are the form factors corresponding to nuclear matrix elements, $M_\alpha^{0\nu}$, given by the equation (5.4.14).

Thus the transition amplitude, $V_{if}^{0\nu}$, from equation (5.4.20) can be written as:

$$V_{fi}^{0\nu}=-\frac{g_A^2 m_e}{2\pi}\left(\frac{G_F}{\sqrt{2}}\right)^2\frac{1}{R_A}\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)M^{0\nu}\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right), \qquad (5.4.23)$$

with the nuclear matrix element, $M^{0\nu}$, given by:

$$M^{0\nu}=M_{GT}^{0\nu}-\left(\frac{g_V}{g_A}\right)^2 M_F^{0\nu}+M_T^{0\nu}.$$

5.5 Separation of the decay rate into nuclear matrix element and phase space factors

We employed the following Fermi-Golden rule for the differential decay rate, $d\Gamma^{0\nu}$, to calculate the half-life, $T_{1/2}^{0\nu}$ of the $0\nu\beta\beta$ decay process:

$$transition\ probability=\frac{2\pi}{\hbar}|transition\ amplitude|^2\times density\ of\ states,$$

where, $\hbar$ is the reduced Planck's constant. In our case, $\hbar=c=1$, c being the speed of light. We have arrived the following formula for the differential decay rate, $d\Gamma^{0\nu}$:

$$d\Gamma^{0\nu}=\left|V_{fi}^{0\nu}\right|^2\frac{1}{2}\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}2\pi\delta(E_1+E_2+E_f-E_i), \qquad (O.2.3)$$

where, $V_{if}^{0\nu}$, is the transition amplitude given by the equation (5.4.23), $\vec{p}_1$, $\vec{p}_2$ and E_1, E_2 are the momenta and energies of emitting electrons respectively, E_i and E_f are the energies of the initial and final nuclear states of the $0\nu\beta\beta$ beta decay process.

By summing over the spins of the emitting electrons using Casimir's trick described in the appendix L and substituting the following expression:

$$\sum_{all\ spins} \left|\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)\right|^2 = 8E_1E_2(1-\alpha cos\theta), \qquad \alpha = \frac{|\vec{p}_1||\vec{p}_2|}{E_1E_2}, \qquad (L.21)$$

into the transition amplitude, $V_{if}^{0\nu}$, given by the equation (5.4.23) we get:

$$\left|V_{fi}^{0\nu}\right|^2 = \frac{2m_e^4G_F^4}{(2\pi)^2}\frac{g_A^4}{r_A^2}E_1E_2(1-\alpha cos\theta)|M^{0\nu}|^2\left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2, \qquad (5.5.1)$$

with, $r_A = m_eR_A$, $R_A = 1.2A^{1/3}$fm, R_A is the radius of the daughter nucleus of mass number, A, m_e is the mass of electron and θ is the angle between the emitting electrons.

Substituting the equation (5.5.1) into the equation $(O.2.3)$, and integrating over the angles between the emitting electrons, θ, and over the energy of electron, E_2, we have the following formula for the differential decay rate of the $0\nu\beta\beta$ decay process:

$$d\Gamma^{0\nu} = \frac{m_e^4G_F^4}{(2\pi)^5}\frac{g_A^4}{r_A^2}p_1'p_2'E_1E_2dE_1|M^{0\nu}|^2\left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2,$$

with

$$p_k' = |\vec{p}_k|, \qquad E_k^2 = p_k'^2 + m_e^2, \qquad k = 1,2. \qquad (5.5.2)$$

Introducing the momenta, $\tilde{p}_1$ and $\tilde{p}_2$ and energies, ε_1 and ε_2 of electrons in the units of electron mass,m_e and including the relativistic Fermi factor, $F_0(Z_f,\varepsilon_k)$, [82] of Coulomb corrections, given by the following expression:

$$F_0(Z_f,\varepsilon_k) = \frac{4}{[\Gamma(2\gamma_1+1)]^2}(2p_k'R_A)^{2(\gamma_1-1)}|\Gamma(\gamma_1+iy_k)|^2e^{\pi y_k}, \qquad k = 1,2,$$

with

$$\gamma_1 = \sqrt{1-(\alpha Z_f)^2}, \qquad y_k = \alpha Z_f\varepsilon_k/p_k', \qquad Z_f = Z_i + 2, \qquad (5.5.3)$$

where, $\alpha = 1/137$, is the fine-structure constant, $\Gamma(x)$, is the gamma function and Z_i and Z_f are the atomic numbers of the initial and final nucleus respectively, we can write down the decay rate, $\Gamma^{0\nu}$ of the of the $0\nu\beta\beta$ beta decay process in the following form by integrating over the energy of electron, ε_1:

$$\Gamma^{0\nu} = \int_1^{T+1} d\Gamma^{0\nu} = \frac{m_e^9 G_F^4}{(2\pi)^5} \frac{g_A^4}{r_A^2} \int_1^{T+1} \left[b_1^{\beta\beta} F_0(Z_f, \varepsilon_1) F_0(Z_f, \varepsilon_2) \tilde{p}_1 \tilde{p}_2 \varepsilon_1 \varepsilon_2 d\varepsilon_1 \right] |M^{0\nu}|^2 \left| \left(\frac{\langle m_{\beta\beta} \rangle}{m_e} \right) \right|^2,$$

with

$$b_1^{\beta\beta} = 1, \varepsilon_k = \frac{E_k}{m_e}, \quad \tilde{p}_k = \frac{p_k'}{m_e} = \frac{|\vec{p}_k|}{m_e} = \sqrt{\varepsilon_k^2 - 1}, \quad \varepsilon_2 = T + 2 - \varepsilon_1, \quad k = 1,2, \tag{5.5.4}$$

where T is the Q-value of the of the $0\nu\beta\beta$ beta decay process in the units of electron mass, m_e.

The relativistic Fermi factor, $F_0(Z_f, \varepsilon_k)$, takes into account the Coulomb attraction between the emitting electrons and the daughter nucleus. It is defined as the square of the ratio of the values of the Dirac s-wave functions of the electron at the nuclear surface (i.e. at radius, $R_A = 1.2A^{1/3}$ fm) with and without the Coulomb potential of a homogeneously charged sphere.

Experimentalists measure the half-life, $T_{1/2}^{0\nu}$, of the $0\nu\beta\beta$ beta decay process which is related to the decay rate, $\Gamma^{0\nu}$, by the following formula:

$$\left[T_{1/2}^{0\nu}\right]^{-1} = \frac{\Gamma^{0\nu}}{ln2}, \tag{5.5.5}$$

Substituting the equation (5.5.4) into the equation (5.5.5), we have finally arrived at the following formula for the half-life, $T_{1/2}^{0\nu}$, of the the $0\nu\beta\beta$ decay process:

$$\left[T_{1/2}^{0\nu}\right]^{-1} = G_1^{0\nu} |M^{0\nu}|^2 \left| \left(\frac{\langle m_{\beta\beta} \rangle}{m_e} \right) \right|^2, \tag{5.5.6}$$

where phase space integral, $G_1^{0\nu}$, is given by:

$$G_1^{0\nu} = \frac{g^{0\nu}}{r_A^2}\int_1^{T+1} b_1^{\beta\beta}\, F_0\big(Z_f, \varepsilon_1\big)F_0\big(Z_f, \varepsilon_2\big)\tilde{p}_1\tilde{p}_2\varepsilon_1\varepsilon_2 d\varepsilon_1,$$

with

$$g^{0\nu} = (m_e^9 G_F^4 g_A^4)/(32\pi^5 ln2) = 2.80 \times 10^{-22} g_A^4\ yr^{-1}. \qquad (5.5.7)$$

CHAPTER VI

SOME NUMERICAL RESULTS OF THE PHASE SPACE FACTORS

Two crucial ingredients to calculate the decay rate, $\Gamma^{0\nu}$ for the $0\nu\beta\beta$ decay process are the nuclear matrix element, $M^{0\nu}$ and the phase space factor, $G_1^{0\nu}$. The phase space factor, $G_1^{0\nu}$ for the $0\nu\beta\beta$ decay process contains the kinematic information about the two final state electrons and is simpler than that of $2\nu\beta\beta$ decay process because of the absence of integration over the neutrino energies. We have only considered the $0_i^+ \to 0_f^+$, ground state to ground state nuclear transition.

The phase space factor, $G_1^{0\nu}$, given in $(5.5.7)$, is exactly calculable to the precision of the input parameters. The input parameters in the calculation of the phase space factor, $G_1^{0\nu}$ are the Q value, $Q_{\beta\beta}$, and the nuclear radius value, R_A of the daughter nuclei. Table 2 shows the most recent results [94] for the phase space factors including electron screening, calculated using the experimental Q values and the nuclear radius estimated from the mass number, A of the daughter nuclei as $R_A = 1.2A^{1/3}$ fm. Details on the uncertainties introduced by this approximation are discussed later. The value of the axial constant, $g_A = 1.254$ is used for the calculation of $G_1^{0\nu}$. Figure 7 shows, $G_1^{0\nu}$ in unit of $g_A^4 \times 10^{-15}\ year^{-1}$ as a function of the mass number, A presented using the values of the input parameters mentioned above.

Table 2. Phase space factors, $G_1^{0\nu}$ obtained using screened exact finite-size Coulomb wave functions for the electrons, calculated by Iachello F. *et al*, *Phys. Rev*. C **85**, 034316 (2012). Q values, $Q_{\beta\beta}$
Taken from the experiments. $R_A = 1.2A^{1/3}$ fm is used for the calculation of the phase space integral.

Nucleus	$Q_{\beta\beta}\ (MeV)$	$G_1^{0\nu}(10^{-15}\ year^{-1})$
${}^{48}_{20}Ca$	4.27226 (404)	61.35
${}^{76}_{32}Ge$	2.03904 (16)	5.843
${}^{82}_{34}Se$	2.99512 (201)	25.12
${}^{96}_{40}Zr$	3.35037 (289)	50.89
${}^{100}_{42}Mo$	3.03440 (17)	39.37
${}^{116}_{48}Cd$	2.81350 (13)	41.29
${}^{128}_{52}Te$	0.86587 (131)	1.453
${}^{130}_{52}Te$	2.52697 (23)	35.16
${}^{136}_{54}Xe$	2.45783 (37)	36.05
${}^{150}_{60}Nd$	3.37138 (20)	155.86

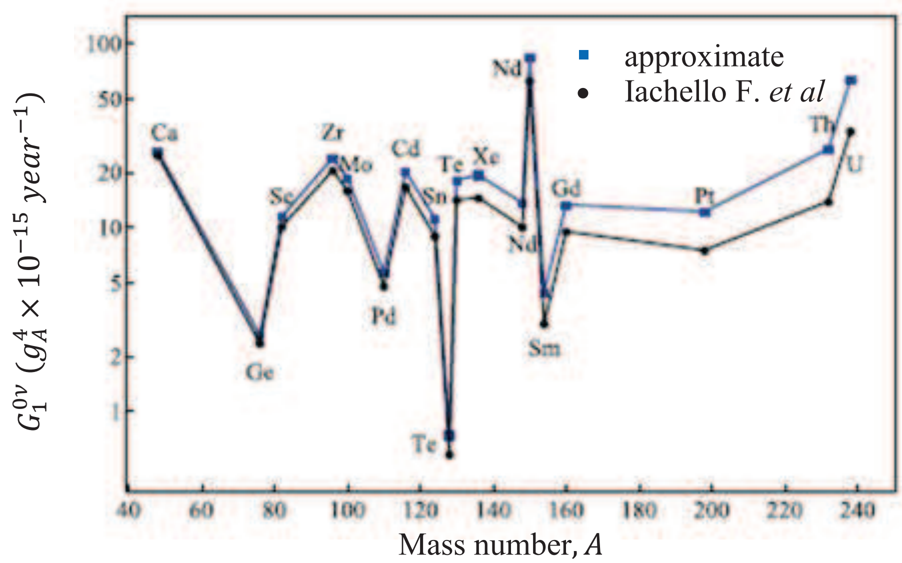

Figure 7. The phase space factors, $G_1^{0\nu}$ in units of $g_A^4 \times 10^{-15}\ year^{-1}$, as a function of the mass number, A.
Calculated by Iachello F. *et al*, *Phys. Rev*. C **85**, 034316 (2012). The input parameters are the same as those used for the calculations of $G_1^{0\nu}$ presented in the table 2. The label "approximate" refers to the results obtained by the use of approximate electron wave functions. The figure is in semi-logarithmic scale.

The phase space factors, $G_1^{0\nu}$ have been calculated by many authors [14, 82, 85, 90, 94, 97–99]. The uncertainties introduced to the calculation of the phase space integral, given in (5.5.7), originate from the errors introduced in the experimental Q values, the errors in estimating the nuclear radius values, R_A and the errors coming from screening of electrons. Recent calculations of phase space factors [94] show that the estimated uncertainty in $G_1^{0\nu}$ due to experimental error of the input parameter, $Q_{\beta\beta}$ is approximately $3 \times \delta\, Q_{\beta\beta}/Q_{\beta\beta}$. For example, recently the Q value for the decay of $^{110}_{46}Pd$ has been measured with high accuracy [95]. Table 3 shows [94] the improvement in the errors in $G_1^{0\nu}$ due to the better accuracy obtained by measurement compared to the $Q_{\beta\beta}$ determined from the mass values.

Table 3. The uncertainty on the phase space factor, $G_1^{0\nu}$ due to the uncertainty of the Q value, $Q_{\beta\beta}$ [94].

References	Nucleus	$Q_{\beta\beta}$ (MeV)	$G_1^{0\nu}$ $(10^{-15}\ year^{-1})$	$\sigma_{G_1^{0\nu}}$	$3 \times \delta\, Q_{\beta\beta}/Q_{\beta\beta}$
[96]	$^{110}_{46}Pd$	2.00400 (1133)	4.707 (86)	1.83 %	1.69 %
[95]	$^{110}_{46}Pd$	2.01785 (64)	4.815 (06)	0.12 %	0.095 %

The nuclear radius, R_A enters in the calculation of phase space integral in two ways: (i) through the scaling factor in the definition of $G_1^{0\nu}$, and (ii) through the relativistic Fermi factor, $F_0(Z_f, \varepsilon_k)$, defined in (5.5.3). The scaling factor $(1/R_A^2)$ of $G_1^{0\nu}$ has no effect in the calculation of the decay rate, $\Gamma^{0\nu}$, since this factor is compensated by a factor (R_A) introduced in the definition of the nuclear matrix element, $M^{0\nu}$, to make $M^{0\nu}$ dimensionless. However, since the relativistic Fermi factor, $F_0(Z_f, \varepsilon_k)$, in the phase space integral (5.5.7), explicitly depends on the nuclear radius, $R_A = r_0 A^{1/3}$, an error is introduced in the calculation of $G_1^{0\nu}$, whenever a choice is made for the value of r_0 in compared to the value of r_0 chosen such that the experimental value of mean square radius, $\langle r^2 \rangle_{exp}$ (whenever available) is reproduced for each nuclei [94]. The choices of the values of r_0 are found to be different in the literature. Many authors have used $r_0 = 1.2$ fm [82, 85, 94], some authors have used $r_0 = 1.1$ fm [90, 97], and others [98] preferred to use the values of r_0 obtained by adjusting the experimental values of $\langle r^2 \rangle_{exp}$ to calculate the phase space factors. In the most recent calculations of $G_1^{0\nu}$, Kotila J. and Iachello F. [94] have reported that assuming a nuclei being a uniformly charged sphere and adjusting r_0 for each nuclei with atomic number, Z and mass number, A using

$$\frac{3}{5} r_0^2 A^{2/3} = \langle r^2 \rangle_{exp}\ , \tag{6.1}$$

where, $\langle r^2 \rangle_{exp}$ is obtained from electron scattering and/or muonic x rays, the largest difference between $(R_A)_{theo} = 1.2A^{1/3}$ fm and $(R_A)_{exp}$ is found to be ~ 4 %. This leads to an estimate of error of 7 % in the calculation of $G_1^{0\nu}$.

In addition, there is also an error introduced to the calculation of $G_1^{0\nu}$ coming from screening of electrons. Tomoda [14] presented the results for the calculation of the phase space factors for a selected number of nuclei obtained by approximating the electron wave functions at the nuclear radius and without inclusion of electron screening. By taking advantage of some recent developments in the numerical evaluation of Dirac wave functions and in the solution of the Thomas-Fermi equation to calculate more accurate phase space factors for the $0\nu\beta\beta$ decay in all nuclei of interest, recent calculations [94] of $G_1^{0\nu}$ show an estimate of the screening error to be 10 % of the Thomas-Fermi contribution, which is known to overestimate the electron density at the nucleus. This gives an error in the calculation of $G_1^{0\nu}$ of 0.10 %. These results are of particular interest for heavy nuclei, where relativistic and screening corrections play a major role. Table 4 summarizes the uncertainties in the calculation of $G_1^{0\nu}$ coming from the different input parameters. A comparison of the phase space factors calculated by different authors [82, 98, 99] with the most recent calculation [94] is presented in figure 8. The differences among the calculated results arise from the fact that different authors use different approximations of the wave functions of the emitting electrons.

Table 4. Estimate of uncertainties [94] introduced to the phase space factor, $G_1^{0\nu}$ due to different input parameters.

Input parameters	Q value, $Q_{\beta\beta}$	Radius, R_A	Electron screening
Uncertainty on $G_1^{0\nu}$	$3 \times \delta\, Q_{\beta\beta}/Q_{\beta\beta}$	7 %	0.10 %

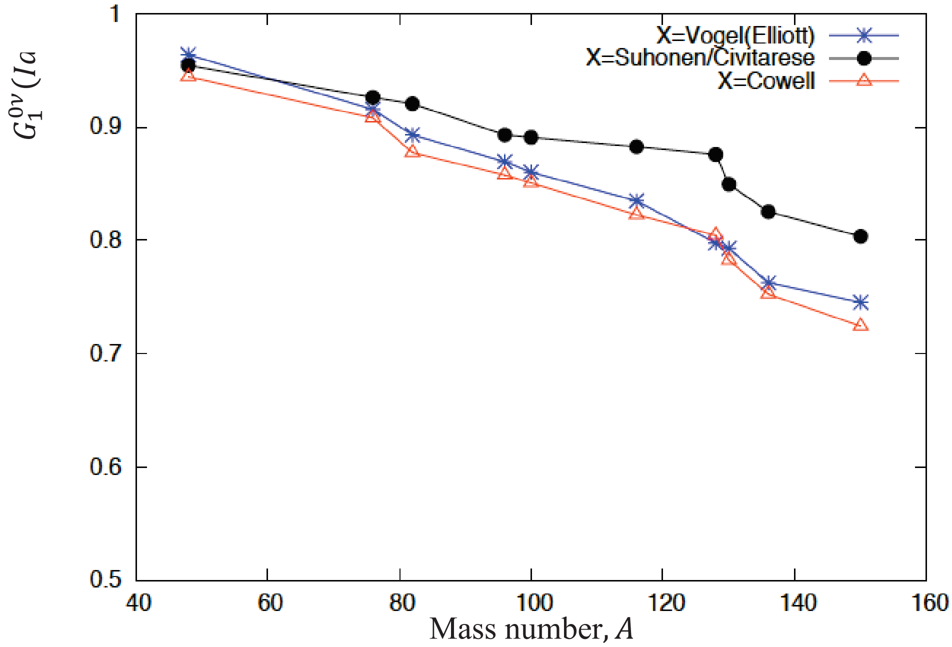

Figure 8. Comparison of the phase space factors, $G_1^{0\nu}$, calculated by the different authors [82, 98, 99] with the results.
Calculated by Iachello F. *et al*, *Phys. Rev*. C **85**, 034316 (2012). The value of the axial constant, $g_A = 1.254$ is used for all of the calculations of $G_1^{0\nu}$. The figure is compiled by Horoi M., CMU.

CHAPTER VII

CONCLUSION

Neutrinoless double beta decay has a long and interesting history with important implications in nuclear and particle physics, astrophysics and cosmology, but its observation is still elusive. We have briefly reviewed the history of double beta decay in chapter II. We have seen that $0\nu\beta\beta$ decay is an exotic process, which is expected to occur if LN conservation is not an exact symmetry of nature. It is thus forbidden in the SM of electroweak interaction. The requirement that the LN conservation principle has to be violated in order for the $0\nu\beta\beta$ decay to occur, led us to search for new physics beyond the SM of particle physics. Thus the study of the $0\nu\beta\beta$ decay process is stimulated by the development of modern grand unified theories (GUTs) and supersymmetric (SUSY) extensions of the SM of particle physics, which suppose that such conservation laws of the SM may be violated to some small degree. SUSY also offer a variety of mechanisms which allow the $0\nu\beta\beta$ decay to occur.

The recent discovery of neutrino oscillations of solar, atmospheric, accelerator and reactor neutrinos has given the first evidence of physics beyond the SM of particle physics and in particular, indicates that the neutrinos are massive particles with masses many orders of magnitude smaller than those of charged leptons. The oscillations were able to show that the neutrinos are admixed, determined two of the mixing angles and set a stringent limit on the third. Furthermore, they have also determined one square mass difference and the absolute value of the other. To further understand neutrinos, we must know whether they are Dirac or Majorana fermions, an issue which only $0\nu\beta\beta$ decay can decide, regardless of the mechanism causing the decay process. Hence the observation of $0\nu\beta\beta$ beta decay process and a measurement of the decay rate, $\Gamma^{0\nu}$ when combined with neutrino oscillation data, in addition to confirming the

Majorana nature of the neutrinos, would give information on the absolute neutrino mass scale, and potentially also on the neutrino mass hierarchy and the Majorana phases appearing in the unitary neutrino mixing matrix. But to achieve these goals we need an accurate evaluation of the nuclear matrix elements, $M^{0\nu}$ that enters in the half-life formula of the decay. Chapter III provides a brief exposure on our current knowledge about neutrino masses and mixings in connection with the $0\nu\beta\beta$ decay, ordinary β decay and cosmology. We have seen that astrophysical and cosmological observations set upper bounds on the sum of neutrino masses and the future β decay experiments will be able to distinguish the IH from the quasidegenerate spectrum if there is a way to measure the mass of the lightest neutrino from those experiments.

In this project we have included relativistic effects and higher order terms (chapter V) in the nucleon current, $J_L^{\mu\dagger}(x)$ for the correct evaluation of the nuclear matrix elements, $M^{0\nu}$ and hence to calculate the decay rate, $\Gamma^{0\nu}$ for the $0\nu\beta\beta$ decay process. Chapter IV provides the necessary Quantum field theoretical background for our purpose. We have started from the second-order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$, since $0\nu\beta\beta$ decay occurs in second order of perturbation theory of weak interaction. Within the framework of the interaction picture, the connection between $\langle f|S^{(2)}|i\rangle^{0\nu}$ and the effective weak β decay Hamiltonian density, $\mathcal{H}_w^{\beta}(x)$, is provided by the so-called Dyson expansion of the S operator. We then applied the Wick's theorem to write down the time ordered product of the leptonic current in terms of the normal ordered product, which means that all the annihilation operators in the leptonic current should be placed to the right of all the creation operators. For the hadronic current, we have made a transition from the interaction picture to the Heisenberg picture, since the strong-interaction Hamiltonian density, $\mathcal{H}_s^{I}(x)$ is taken into account in the interaction picture and a perturbative treatment of which is not possible within the framework of interaction picture. Assuming neutrinos are Majorana particles and neutrino mixing does take place, applying the quantization technique for the Dirac fields of electrons, and evaluating the time ordered product of the hadronic current by splitting the hadronic current into space and time components, we have obtained an expression for the $\langle f|S^{(2)}|i\rangle^{0\nu}$. We have further made two approximations, namely, the closure and the long wave approximations (see section 5.4) for the second-order perturbative matrix element.

Next, we have included the higher order terms (see section 5.4) in the nucleon current, $J_L^{\mu\dagger}(x)$, namely, the pseudoscalar and the weak-magnetism terms, which are beyond the familiar vector-axial vector structure of the electroweak Lagrangian. We have assumed the light-neutrino exchange throughout and the impulse approximation, i.e., nucleons in nuclei can be treated as free when dealing with the weak interaction. For nuclear structure calculations, it is necessary to reduce the nucleon current to the nonrelativistic form. For this purpose, we have neglected small energy transfers between nucleons in the nonrelativistic expansion of the nucleon current. Then the form of the nucleon current, we have obtained, coincides with those in the so-called Breit frame.

With the further approximation that the masses, m_k of the neutrino mass eigenstates can be neglected in compared to the magnitude of the momentum transfer, $|\vec{q}|$ from hadrons to leptons, we have extracted the transition amplitude, $V_{fi}^{0\nu}$ from $\langle f|S^{(2)}|i\rangle^{0\nu}$ using perturbation theory. The transition amplitude, $V_{fi}^{0\nu}$ contains Gamow-Teller, Fermi and Tensor nuclear matrix elements and is an essential ingredient to calculate the decay rate , $\Gamma^{0\nu}$. The Gamow-Teller, $M_{GT}^{0\nu}$

and the Tensor, $M_T^{0\nu}$ components of the nuclear matrix elements, $M^{0\nu}$ contain the higher order contributions of the nucleon current.

Finally, we have employed the non-relativistic Fermi's Golden rule to calculate the decay rate, $\Gamma^{0\nu}$, since the hadronic transition is nonrelativistic. We have assumed the impulse approximation both for the parent and daughter nuclei and hence there is no phase space contribution coming from the hadronic fields (see section 5.5 and the appendix O). However, the states corresponding to the final electrons are described by the relativistic quantum fields. Thus, to include the relativistic effects of the leptonic fields through the density of states, we have introduced the covariant normalization of the free Dirac fields of electrons. By summing over the spins of the emitting electrons using Casimir's trick, including the relativistic Fermi factor, $F_0(Z_f, \varepsilon_k)$, which takes into account the Coulomb distortion of the emitting electrons and the daughter nucleus and integrating over all outgoing momenta of electrons, we have obtained our desired expression of the half-life, $T_{1/2}^{0\nu}$ of the $0\nu\beta\beta$ decay process in terms of the nuclear matrix elements, $M^{0\nu}$, the phase space factor, $G_1^{0\nu}$ and the effective neutrino mass, $\langle m_{\beta\beta}\rangle$. In chapter VI we have presented the most recent results of the phase space factors, $G_1^{0\nu}$ that contain the kinematic information about the two final state electrons of the $0\nu\beta\beta$ decay process. We have also provided a comparison of the phase space factors, $G_1^{0\nu}$, calculated by the different authors with the most recent results.

APPENDICES

APPENDIX A

PROPERTIES OF THE PAULI AND DIRAC GAMMA MATRICES

A.1. Pauli matrices

The components of the Pauli spin vector, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, are given by:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1.1}$$

which satisfy the following relations:

$$\sigma^i \sigma^j = \delta_{ij} + i\epsilon_{ijk}\sigma^k, \qquad (\sigma^i)^\dagger = \sigma^i = (\sigma^i)^{-1}, \qquad i, j, k = 1,2,3, \tag{A.1.2}$$

where, δ_{ij}, is the Kronecker delta and, ϵ_{ijk}, is the antisymmetric tensor, given by:

$$\epsilon_{ijk} = \begin{cases} 1, & if\ ijk = 123,\ \ 231,\ \ 312 \\ -1, & if\ ijk = 132,\ \ 213,\ \ 321 \\ 0, & otherwise \end{cases} . \tag{A.1.3}$$

The commutation and anti-commutation relations of Pauli spin matrices follow from $(A.1.2)$:

$$[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k, \qquad \{\sigma^i, \sigma^j\} = 2\delta_{ij}. \tag{A.1.4}$$

A.2. Dirac gamma matrices

In Pauli-Dirac representation, the four contravariant Dirac gamma matrices, γ^μ, are given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} \mathbb{0} & \sigma^k \\ -\sigma^k & \mathbb{0} \end{pmatrix}, \qquad k = 1,2,3, \tag{A.2.1}$$

where, $\mathbb{1}$, $\mathbb{0}$, denote 2×2 unit and zero matrices, respectively, and, σ^k 's are Pauli matrices, given by $(A.1.1)$. The Dirac gamma matrices satisfy the following anti-commutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad \mu, \nu = 1,2,3,4, \tag{A.2.2}$$

where, $g^{\mu\nu} = diag(1, -1, -1, -1)$, is the metric tensor. The Hermitian conjugations of Dirac gamma matrices are given by:

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^k)^\dagger = -\gamma^k, \qquad k = 1,2,3. \qquad (A.2.3)$$

Using anti-commutation relations, $(A.2.2)$, the hermitian conjugation results given by, $(A.2.3)$, can be summarized by:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \qquad \mu = 1,2,3,4. \qquad (A.2.4)$$

Important combinations of the Dirac gamma matrices are the fifth gamma matrix, γ^5, and the six-component antisymmetric tensor, $\sigma^{\mu\nu}$, defined by:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \qquad \mu, \nu = 1,2,3,4. \qquad (A.2.5)$$

In Pauli-Dirac basis, γ^5, and, $\sigma^{\mu\nu}$, have the following representations:

$$\gamma^5 = \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{pmatrix}, \qquad \sigma^{ij} = \begin{pmatrix} \sigma^k & \mathbb{0} \\ \mathbb{0} & \sigma^k \end{pmatrix}\epsilon_{ijk}, \qquad \sigma^{0k} = i\begin{pmatrix} \mathbb{0} & \sigma^k \\ \sigma^k & \mathbb{0} \end{pmatrix}, \qquad i, j, k = 1,2,3. \qquad (A.2.6)$$

The fifth gamma matrix, γ^5, anti-commutes with all other Dirac gamma matrices:

$$\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0, \qquad \mu = 1,2,3,4. \qquad (A.2.7)$$

γ^5 satisfies the following relations:

$$(\gamma^5)^\dagger = \gamma^5, \qquad (\gamma^5)^2 = 1. \qquad (A.2.8)$$

where, 1, denotes 4×4 unit matrix. The left and right handed projection operators are defined in terms of the fifth gamma matrix, γ^5, by the following:

$$P_L = \left(\frac{1 - \gamma^5}{2}\right), \qquad P_R = \left(\frac{1 + \gamma^5}{2}\right), \qquad (A.2.9)$$

The operators, P_L, and, P_R, have the appropriate properties to be projection operators, that is,

$$(P_i)^2 = P_i, \qquad P_L + P_R = 1, \qquad P_L P_R = P_R P_L = 0, \qquad i = L, R\,. \qquad (A.2.10)$$

APPENDIX B

SPINOR TECHNOLOGY

The four- component free Dirac spinors for particles, $u(p,s)$ and for antiparticles, $v(p,s)$ can be written as:

$$u(p,s) = N\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\chi_s \end{pmatrix}, \qquad v(p,s) = N(-)^{1/2-s}\begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{E+m}\chi_{-s} \\ \chi_{-s} \end{pmatrix}, \qquad s = \pm\frac{1}{2}. \qquad (B.1)$$

Here, E is the energy, p is the four-momentum of a Dirac Particle of mass, m. χ_s 's are two-component Pauli spinors with $\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. s denotes the spin projection. The normalization constant, N is given by:

$$N = \begin{cases} (E+m)^{1/2} & for\ relativistic\ spinors \\ \left(\frac{E+m}{2m}\right)^{1/2} & for\ nonrelativistic\ spinors \end{cases}. \qquad (B.2)$$

The spinors, $u(p,s)$ and $v(p,s)$ satisfy the momentum space Dirac equations:

$$(\gamma^\mu p_\mu - m)u(p,s) = 0, \qquad (\gamma^\mu p_\mu + m)v(p,s) = 0. \qquad (B.3)$$

The adjoint spinors, $\overline{u}(p,s)$ and $\overline{v}(p,s)$ corresponding to the spinors, $u(p,s)$ and $v(p,s)$, respectively, defined by:

$$\overline{u}(p,s) = u^\dagger(p,s)\gamma^0, \qquad \overline{v}(p,s) = v^\dagger(p,s)\gamma^0, \qquad (B.4)$$

satisfy

$$\overline{u}(p,s)(\gamma^\mu p_\mu - m) = 0, \qquad \overline{v}(p,s)(\gamma^\mu p_\mu + m) = 0. \qquad (B.5)$$

The orthonormality conditions of the spinors are given by the following:

$$\overline{u}(p,s')u(p,s) = 2m\delta_{s's}, \qquad \overline{v}(p,s')v(p,s) = -2m\delta_{s's},$$

$$\overline{u}(p,s')v(p,s) = 0, \qquad \overline{v}(p,s')u(p,s) = 0. \qquad (B.6)$$

The spinors, $u(p,s)$ and $v(p,s)$ satisfy the completeness relation in the sense that:

$$\sum_{s=\frac{1}{2},-\frac{1}{2}} u(p,s)\bar{u}(p,s) = \gamma^{\mu}p_{\mu} + m, \qquad \sum_{s=\frac{1}{2},-\frac{1}{2}} v(p,s)\bar{v}(p,s) = \gamma^{\mu}p_{\mu} - m. \qquad (B.7)$$

In terms of the left and right handed projection operators, P_L and P_R, defined in $(A.2.9)$, the left and right handed chiral spinors, both for particles and antiparticles, are defined by:

$$u_L(p,s) = \left(\frac{1-\gamma^5}{2}\right)u(p,s), \qquad u_R(p,s) = \left(\frac{1+\gamma^5}{2}\right)u(p,s),$$

$$v_L(p,s) = \left(\frac{1+\gamma^5}{2}\right)v(p,s), \qquad v_R(p,s) = \left(\frac{1-\gamma^5}{2}\right)v(p,s), \qquad (B.8)$$

where the subscript, 'L' and 'R' stand for 'left handed' and 'right handed', respectively. Using the definition of adjoint spinors, $\bar{u}(p,s)$ and $\bar{v}(p,s)$ given in $(B.4)$, the corresponding left and right handed adjoint spinors, both for particles and antiparticles, are given by:

$$\bar{u}_L(p,s) = \bar{u}(p,s)\left(\frac{1+\gamma^5}{2}\right), \qquad \bar{u}_R(p,s) = \bar{u}(p,s)\left(\frac{1-\gamma^5}{2}\right),$$

$$\bar{v}_L(p,s) = \bar{v}(p,s)\left(\frac{1-\gamma^5}{2}\right), \qquad \bar{v}_R(p,s) = \bar{v}(p,s)\left(\frac{1+\gamma^5}{2}\right), \qquad (B.9)$$

since, γ^5 is hermitian, that is, $(\gamma^5)^{\dagger} = \gamma^5$, and γ^5 anti-commutes with γ^0.

APPENDIX C

CHARGE CONJUGATION AND MAJORANA CONDITION

An important discrete symmetry transformation to spinor field is charge conjugation. It has nothing to do with the space or time coordinates, instead, it is related to the particle-antiparticle degree of freedom of a theory. Charge conjugation exchanges the roles of particle and antiparticle spinors. For Dirac field, $\psi(x)$, this exchange is mediated by the following transformation:

$$\psi(x) \rightarrow \psi_c(x) = C\gamma^0\psi^*(x) = C\overline{\psi}^T, \qquad (C.1)$$

where '$*$' and T denote complex conjugate and transpose, respectively. C is the charge conjugation matrix. The fields, $\psi(x)$ and $\psi_c(x)$, both satisfy the Dirac equation but their coupling to the electromagnetic field involves opposite sign of the elementary charge. The charge conjugation matrix, C is constructed such a way that it has the following properties:

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu, \qquad C^T = C^\dagger = C^{-1} = -C. \qquad (C.2)$$

Using the definition of the fifth gamma matrix, γ^5, given by, $(A.2.5)$, its follows form $(C.2)$ and the anti-commutation relations of the Dirac gamma matrices, given by, $(A.2.2)$, that, under charge conjugation, γ^5 satisfy the following condition:

$$C(\gamma^5)^T C^{-1} = C(i\gamma^0\gamma^1\gamma^2\gamma^3)^T C^{-1} = iC(\gamma^3)^T C^{-1} C(\gamma^2)^T C^{-1} C(\gamma^1)^T C^{-1} C(\gamma^0)^T C^{-1}$$

$$= i(-\gamma^3)(-\gamma^2)(-\gamma^1)(-\gamma^0) = i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \qquad (C.3)$$

Within the standard Pauli-Dirac representation of the Dirac gamma matrices, γ^μ, are given by $(A.2.1)$, the charge conjugation matrix, C is given by:

$$C = i\gamma^2\gamma^0 = -i\begin{pmatrix} \mathbb{0} & \sigma^2 \\ \sigma^2 & \mathbb{0} \end{pmatrix}, \qquad (C.4)$$

where, $\mathbb{0}$, denotes 2×2 zero matrix, and, σ^2 is the Pauli matrix, σ^k with $k = 2$, given by $(A.1.1)$. Under charge conjugation, the four- component free Dirac spinors for particles, $u(p,s)$ and for antiparticles, $v(p,s)$, given by, $(B.1)$, are interchanged:

$$C\overline{u}^T(p,s) = v(p,s), \qquad C\overline{v}^T(p,s) = u(p,s), \tag{C.5}$$

where C is chosen in the chiral representation, given by $(C.4)$.

Since a Majorana fermion is said to be its own anti-particle up to a phase, a Majorana fermion is a Dirac fermion that is self-conjugate, i.e., a Majorana field, $\chi(x)$ satisfies the following condition, known as Majorana condition:

$$\chi_c(x) = C\overline{\chi}^T(x) = \xi\chi(x), \tag{C.6}$$

where ξ is Majorana phase factor, chosen to be ± 1.

An essential distinction between a Majorana field, $\chi(x)$ and a Dirac field, $\psi(x)$ is that for a Majorana field, $\chi(x)$ the vector current, $\overline{\chi}(x)\gamma^\mu\chi(x)$, is identically zero. This can be proved as follows:

$$\overline{\chi}(x)\gamma^\mu\chi(x) = \left(\overline{\chi}(x)\gamma^\mu\chi(x)\right)^T = -\chi^T(x)(\gamma^\mu)^T\overline{\chi}^T(x). \tag{C.7}$$

Here we have used the fact that, as the quantity, $\overline{\chi}(x)\gamma^\mu\chi(x)$ is a 1×1 matrix, thus we can write it as its transpose. From the Majorana condition $(C.6)$, we can write:

$$\overline{\chi}^T(x) = \xi C^{-1}\chi(x), \tag{C.8}$$

and

$$\chi^T(x) = \frac{1}{\xi}\left(C\overline{\chi}^T(x)\right)^T = \frac{1}{\xi}\,\overline{\chi}(x)C^T = -\xi\overline{\chi}(x)C, \tag{C.9}$$

since $\xi = \pm 1$ and $C^T = -C$. Substituting $(C.8)$ and $(C.9)$ into $(C.7)$, we have:

$$\overline{\chi}(x)\gamma^\mu\chi(x) = \xi^2\overline{\chi}(x)C(\gamma^\mu)^T C^{-1}\chi(x) = -\overline{\chi}(x)\gamma^\mu\chi(x), \tag{C.10}$$

since, $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$ and $\xi^2 = 1$.

Thus $(C.10)$ implies:

$$\overline{\chi}(x)\gamma^{\mu}\chi(x) = 0. \qquad (C.11)$$

The origin of the extra minus sign, introduced in $(C.7)$ is subtle but important. It is clearly necessary for a physically meaningful result. The minus sign is related to the connection between spin and statistics in the quantum field theory. It occurs because of the antisymmetric nature of fermion fields, i.e., the minus sign arises due to the exchange of the order of the two fermion fields.

APPENDIX D

CANONICAL QUANTIZATION OF FREE SPINOR FIELDS

D.1. Canonical quantization of Dirac field

The Lagrangian, $\mathcal{L}^{Dirac}$ of the free Dirac field, $\psi(x)$, describing a spin-1/2 particle of mass, m, is given by [79]:

$$\mathcal{L}^{Dirac} = i\overline{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x) - m\overline{\psi}(x)\psi(x). \qquad (D.1.1)$$

The field, $\psi(x)$ obeys the Dirac equation:

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0. \qquad (D.1.2)$$

The general solutions of the Dirac equation $(D.1.2)$ can be written as:

$$\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E}\left[b(p,s)u(p,s)e^{-ipx} + d^{\dagger}(p,s)v(p,s)e^{ipx}\right], \qquad (D.1.3)$$

where $u(p,s)$ and $v(p,s)$ are the four- component free Dirac spinors for particles and for antiparticles with four-momentum, p and a definite spin projection, s respectively, given in $(B.1)$. Here, E is the energy and $\vec{p}$ is the momentum of a Dirac Particle of mass, m. The operators $b^{\dagger}(p,s)$ and $b(p,s)$ create and annihilate a Dirac particle of given spin, s and four momentum, p, while the operators $d^{\dagger}(p,s)$ and $d(p,s)$ create and annihilate an antiparticle of given spin, s and four momentum, p.

The free Dirac field, $\psi(x)$, being a spinor field, obeys the following equal time anticommutation relations [79]:

$$\left\{\psi_{\alpha}(t,\vec{x}), \psi_{\beta}(t,\vec{y})\right\} = 0, \qquad \left\{\psi_{\alpha}(t,\vec{x}), \overline{\psi}_{\beta}(t,\vec{y})\right\} = (\gamma^0)_{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \qquad (D.1.4)$$

The anticommutation relations $(D.1.4)$ led us the following anticommutation relations for the creation and annihilation operators for the free Dirac field [79]:

$$\{b(p,s), b(p',s')\} = \{b^\dagger(p,s), b^\dagger(p',s')\} = 0,$$

$$\{d(p,s), d(p',s')\} = \{d^\dagger(p,s), d^\dagger(p',s')\} = 0,$$

$$\{b(p,s), d(p',s')\} = \{b^\dagger(p,s), d^\dagger(p',s')\} = \{b(p,s), d^\dagger(p',s')\} = \{b^\dagger(p,s), d(p',s')\} = 0,$$

$$\{b(p,s), b^\dagger(p',s')\} = \{d^\dagger(p,s), d(p',s')\} = (2\pi)^3 2E\delta^3(\vec{p}-\vec{p}')\delta_{ss'}. \quad (D.1.5)$$

D.2. Canonical quantization of Majorana field

Since a Majorana fermion is a Dirac fermion, which is identical with its own anti-particle up to a phase, the equation of motion for the Majorana field, $\chi(x)$, is the Dirac equation, that is, the free Majorana field, $\chi(x)$ obeys:

$$\left(i\gamma^\mu \partial_\mu - m\right)\chi(x) = 0. \quad (D.2.1)$$

The general solution of $(D.2.1)$ is once again given by $(D.1.3)$:

$$\chi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \left[b(p,s)u(p,s)e^{-ipx} + d^\dagger(p,s)v(p,s)e^{ipx}\right]. \quad (D.2.2)$$

However, the Majorana field, $\chi(x)$ obeys the Majorana condition, $C\overline{\chi}^T(x) = \xi\chi(x)$, given in $(C.6)$, where C is the charge conjugation matrix, T denotes the transpose and ξ is the Majorana phase factor. Since, $\overline{\chi}(x) = \chi^\dagger(x)\gamma^0$, we have

$$\overline{\chi}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \left[b^\dagger(p,s)\overline{u}(p,s)e^{ipx} + d(p,s)\overline{v}(p,s)e^{-ipx}\right], \quad (D.2.3)$$

and thus

$$C\overline{\chi}^T(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \left[b^\dagger(p,s)C\overline{u}^T(p,s)e^{ipx} + d(p,s)C\overline{v}^T(p,s)e^{-ipx}\right]. \quad (D.2.4)$$

From appendix C we have

$$C\overline{u}^T(p,s) = v(p,s), \qquad C\overline{v}^T(p,s) = u(p,s). \quad (C.5)$$

Thus $(D.2.4)$ can be written as:

$$C\overline{\chi}^T(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \left[b^\dagger(p,s)v(p,s)e^{ipx} + d(p,s)u(p,s)e^{-ipx}\right]. \qquad (D.2.5)$$

Now from $(D.2.2)$ and $(D.2.5)$, we see that, the Majorana condition, $C\overline{\chi}^T(x) = \xi\chi(x)$, will hold if

$$d(p,s) = \xi b(p,s), \qquad |\xi|^2 = 1. \qquad (D.2.6)$$

We choose the Majorana phase factor, ξ to be real, thus, $\xi = \pm 1$. Substituting $(D.2.6)$, into $(D.2.2)$, the free Majorana field, $\chi(x)$, can be written as:

$$\chi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \left[b(p,s)u(p,s)e^{-ipx} + \xi b^\dagger(p,s)v(p,s)e^{ipx}\right], \qquad \xi = \pm 1 \qquad (D.2.7)$$

For the free Majorana field, $\chi(x)$, the creation and annihilation operators obey the following anticommutation relations [79]:

$$\{b(p,s), b(p',s')\} = \{b^\dagger(p,s), b^\dagger(p',s')\} = \{b(p,s), b^\dagger(p',s')\} = 0,$$

$$\{b(p,s), b^\dagger(p',s')\} = (2\pi)^3 2E\delta^3(\vec{p} - \vec{p}')\delta_{ss'}. \qquad (D.2.8)$$

APPENDIX E

FEYNMAN DIAGRAM AND PROPAGATORS FOR THE FREE FERMIONIC FIELD

E.1. Propagator in coordinate and momentum space representation

The Feynman propagator is defined as the vacuum expectation value of the time-ordered product of the fields taken at different points of space-time, x. For the free Dirac field, $\psi(x)$, the Feynman propagator is can be written as:

$$\overbrace{\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)} = S_{\alpha\beta}^{Dirac}(x_1 - x_2) = \left\langle 0 \middle| T\left(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\right) \middle| 0 \right\rangle, \qquad (E.1.1)$$

where, T, denotes the time-ordered product, defined by:

$$T\left(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\right) = \theta(t_1 - t_2)\psi_\alpha(x_1)\bar{\psi}_\beta(x_2) - \theta(t_2 - t_1)\bar{\psi}_\beta(x_2)\psi_\alpha(x_1), \qquad (E.1.2)$$

and, $\theta(t)$, is the unit step function, defined by:

$$\theta(t_1 - t_2) = \begin{cases} 1 & for \quad t_1 > t_2 \\ 0 & for \quad t_2 > t_1 \end{cases}. \qquad (E.1.3)$$

Note the minus sign in the second term of $(E.1.2)$ accounts for the fermionic character of the fields, that is,

$$\psi_\alpha(x_1)\bar{\psi}_\beta(x_2) = -\bar{\psi}_\beta(x_2)\psi_\alpha(x_1), \qquad t_1 \neq t_2. \qquad (E.1.4)$$

Since, the Fourier expansions of canonically quantized free Dirac field, $\psi(x)$, and the Majorana field, $\chi(x)$, are given by $(D.1.3)$ and $(D.2.2)$, respectively,

$$\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \left[b(p,s)u(p,s)e^{-ipx} + d^\dagger(p,s)v(p,s)e^{ipx}\right], \qquad (D.1.3)$$

and

$$\chi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \left[b(p,s)u(p,s)e^{-ipx} + \xi b^\dagger(p,s)v(p,s)e^{ipx}\right], \qquad \xi = \pm 1, (D.2.2)$$

it is easy to see that the Feynman propagator, $S^{Majorana}_{\alpha\beta}(x_1 - x_2)$, for the free Majorana field, $\chi(x)$, given by:

$$\overbrace{\chi_\alpha(x_1)\bar{\chi}_\beta(x_2)} = S^{Majorana}_{\alpha\beta}(x_1 - x_2) = \left\langle 0 \middle| T\left(\chi_\alpha(x_1)\bar{\chi}_\beta(x_2)\right) \middle| 0 \right\rangle, \tag{E.1.5}$$

is the same as the Feynman propagator, $S^{Dirac}_{\alpha\beta}(x_1 - x_2)$, for the free Dirac field, $\psi(x)$, given by $(E.1.1)$. For the free fermionic field (both the Dirac and Majorana fields), the Feynman propagator in coordinate space representation is given by the following:

$$S_{\alpha\beta}(x_1 - x_2) = i\int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1 - x_2)}}{q^2 - m^2}\left(\gamma^\mu q_\mu + m\right)_{\alpha\beta}, \tag{E.1.6}$$

In momentum space representation, the Feynman propagator becomes:

$$S_{\alpha\beta}(q) = i\frac{\left(\gamma^\mu q_\mu + m\right)_{\alpha\beta}}{q^2 - m^2}, \tag{E.1.7}$$

Since a Majorana fermion is said to be its own anti-particle up to a phase, it doesn't have the same notion of fermion number (charge) conservation that a Dirac fermion does. The essential distinction between a Majorana field, $\chi(x)$ and a Dirac field, $\psi(x)$ is the fact that for a Majorana field, the propagators,

$$\overbrace{\chi(x_1)\chi^T(x_2)}, \qquad \overbrace{\bar{\chi}^T(x_1)\bar{\chi}(x_2)}, \tag{E.1.8}$$

both violate the conservation of fermion number and no longer vanish as they do for the Dirac case, i.e.,

$$\overbrace{\psi(x_1)\psi^T(x_2)} = 0, \qquad \overbrace{\bar{\psi}^T(x_1)\bar{\psi}(x_2)} = 0. \tag{E.1.9}$$

This can be proved as follows. Since, from the Majorana condition, $C\overline{\chi}^T(x) = \xi\chi(x)$, given in $(C.6)$, for the Majorana field, $\chi(x)$, we have:

$$\overline{\chi}^T(x) = \xi C^{-1}\chi(x), \tag{C.8}$$

and

$$\chi^T(x) == -\xi\overline{\chi}(x)C, \tag{C.9}$$

thus. the propagators in $(E.1.8)$, can be written as:

$$\overbrace{\chi(x_1)\chi^T(x_2)} = -\xi\,\overbrace{\chi(x_1)\overline{\chi}(x_2)}\,C = -\xi S(x_1 - x_2)C, \tag{E.1.10}$$

and

$$\overbrace{\overline{\chi}^T(x_1)\overline{\chi}(x_2)} = \xi C^{-1}\,\overbrace{\chi(x_1)\overline{\chi}(x_2)} = \xi C^{-1}S(x_1 - x_2). \tag{E.1.11}$$

where, the Feynman propagator , $S(x_1 - x_2)$ for the Majorana field, $\chi(x)$ is given by $(E.1.6)$.

E.2. Feynman propagator for the free electron-neutrino field

Due to neutrino mixing, the electron-neutrino field, $\nu_{eL}(x)$ can be written as a linear combination of the fields, $\chi_{kL}(x)$ of the Majorana neutrino mass eigenstates, with mass, m_k:

$$\nu_{eL}(x) = \sum_k U_{ek}^L\chi_{kL}(x) = \sum_k U_{ek}^L\left(\frac{1-\gamma^5}{2}\right)\chi_k(x), \tag{5.1.2}$$

where U_{ek}^L is the first row of the unitary neutrino mixing matrix, U_{PMNS}, given by, (3.2.2). The fields $\chi_k(x)$ satisfy the Majorana condition, given by:

$$C\overline{\chi}_k^T(x) = \xi_k\chi_k(x), \qquad \xi_k = \pm 1, \tag{5.1.3}$$

where, ξ_k's are Majorana phase factors. Transposing $(5.1.2)$, we have:

$$\nu_{eL}^T(x) = \sum_{k\prime} U_{ek\prime}^L\chi_{k\prime L}^T(x) = \sum_{k\prime} U_{ek\prime}^L\,\chi_{k\prime}^T(x)\left(\frac{1-\gamma^5}{2}\right)^T. \tag{E.2.1}$$

Using the expression $(5.1.2)$ and $(E.2.1)$, the Feynman propagator , $\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}$, for the free electron-neutrino field, $\nu_{eL}(x)$, can be written as:

$$\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)} = \sum_{k,k\prime} U_{ek}^L\, U_{ek\prime}^L \left(\frac{1-\gamma^5}{2}\right) \overbrace{\chi_k(x_1)\chi_{k\prime}^T(x_2)} \left(\frac{1-\gamma^5}{2}\right)^T$$

$$= \sum_{k,k\prime} U_{ek}^L\, U_{ek\prime}^L \delta_{kk\prime} \left(\frac{1-\gamma^5}{2}\right) \overbrace{\chi_k(x_1)\chi_k^T(x_2)} \left(\frac{1-\gamma^5}{2}\right)^T$$

$$= -\sum_k (U_{ek}^L)^2\, \xi_k \left(\frac{1-\gamma^5}{2}\right) S_k(x_1 - x_2) C \left(\frac{1-\gamma^5}{2}\right)^T$$

$$= -\sum_k (U_{ek}^L)^2\, \xi_k \left(\frac{1-\gamma^5}{2}\right) S_k(x_1 - x_2) \left(\frac{1-\gamma^5}{2}\right) C, \qquad (E.2.2)$$

since, from $(E.1.10)$, we have, $\overbrace{\chi_k(x_1)\chi_k^T(x_2)} = -\xi_k S_k(x_1 - x_2)C,$ and from $(C.3)$, we have, $C(\gamma^5)^T = (\gamma^5)^T C$. Here, $S_k(x_1 - x_2)$ is the Feynman propagator for the free Majorana neutrino field $\chi_k(x)$ with mass, m_k, given according to $(E.1.6)$:

$$S_k(x_1 - x_2) = i \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x_1 - x_2)}}{q^2 - m_k^2} \left(\gamma^\mu q_\mu + m_k\right). \qquad (E.2.3)$$

Now, since,

$$\left(\frac{1-\gamma^5}{2}\right) \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) = \gamma^\mu \left(\frac{1+\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right) = 0, \qquad (E.2.4)$$

and

$$\left(\frac{1-\gamma^5}{2}\right)^2 = \left(\frac{1-\gamma^5}{2}\right), \qquad (E.2.5)$$

we have,

$$\left(\frac{1-\gamma^5}{2}\right) S_k(x_1 - x_2) \left(\frac{1-\gamma^5}{2}\right)$$

$$= i \left(\frac{1-\gamma^5}{2}\right) \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x_1 - x_2)}}{q^2 - m_k^2} \left(\gamma^\mu q_\mu + m_k\right) \left(\frac{1-\gamma^5}{2}\right)$$

$$= im_k \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x_1-x_2)}}{q^2 - m_k^2} \left(\frac{1-\gamma^5}{2}\right). \qquad (E.2.6)$$

Substituting the expression $(E.2.6)$ into $(E.2.2)$, the Feynman propagator for the free electron-neutrino field, $\nu_{eL}(x)$, can be written as:

$$\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)} = -i \sum_k (U_{ek}^L)^2\, m_k \xi_k \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x_1-x_2)}}{q^2 - m_k^2} \left(\frac{1-\gamma^5}{2}\right) C. \qquad (E.2.7)$$

APPENDIX F

CALCULATION OF THE NORMAL ORDER PRODUCT ON THE LEPTONIC CURRENT USING FIELD QUANTIZATION TECHNIQUE

In this section we will calculate the Normal ordered product on the leptonic current, that is, we will calculate the following quantity:

$$\left\langle p_2, s_2; p_1, s_1 \middle| N\left[\bar{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}\,(\gamma^\nu)^T \bar{e}_L^T(x_2)\right] \middle| 0 \right\rangle, \qquad (F.1)$$

where $e(x)$ and $\nu_{eL}(x)$ represents the field operators for electron and left-handed electron-neutrino, respectively. $|0\rangle$ and $|p_2, s_2; p_1, s_1\rangle$ represents the vacuum state and the two-particle state for electrons with four-momentum, p_1, p_2 and spin projections, s_1, s_2 ,respectively, $\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}$ denotes the Feynman propagator for the free electron-neutrino field, given in $(E.2.7)$. N denotes the normal ordering of field operators, meaning all annihilation operators should be placed to the right of all creation operators.

Using equation $(E.2.7)$, we can write:

$$\bar{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}\,(\gamma^\nu)^T \bar{e}_L^T(x_2)$$

$$= -i\sum_k (U_{ek}^L)^2\, m_k \xi_k \bar{e}_L(x_1)\gamma^\mu \left(\frac{1-\gamma^5}{2}\right) C(\gamma^\nu)^T \bar{e}_L^T(x_2) \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x_1-x_2)}}{q^2 - m_k^2}, \qquad (F.2)$$

and

$$\bar{e}_L(x_1)\gamma^\mu \left(\frac{1-\gamma^5}{2}\right) C(\gamma^\nu)^T \bar{e}_L^T(x_2) = -\bar{e}_L(x_1)\gamma^\mu \left(\frac{1-\gamma^5}{2}\right)\gamma^\nu C \bar{e}_L^T(x_2)$$

$$= -\bar{e}(x_1)\left(\frac{1+\gamma^5}{2}\right)\gamma^\mu \left(\frac{1-\gamma^5}{2}\right)\gamma^\nu C \left(\frac{1+\gamma^5}{2}\right)^T \bar{e}^T(x_2)$$

$$= -\bar{e}(x_1)\gamma^\mu \left(\frac{1-\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right)\gamma^\nu \left(\frac{1+\gamma^5}{2}\right) C \bar{e}^T(x_2)$$

$$= -\overline{e}(x_1)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right)\gamma^\nu C\overline{e}^T(x_2) = -\overline{e}(x_1)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\gamma^\nu C\overline{e}^T(x_2), \qquad (F.3)$$

since,

$$C(\gamma^\nu)^T C^{-1} = -\gamma^\nu, \qquad \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu, \qquad \left(\frac{1-\gamma^5}{2}\right)^2 = \left(\frac{1-\gamma^5}{2}\right).$$

Thus $(F.2)$ can be written as:

$$\overline{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T\overline{e}_L^T(x_2)$$

$$= i\sum_k (U_{ek}^L)^2\, m_k\xi_k\overline{e}(x_1)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\gamma^\nu C\overline{e}^T(x_2)\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}. \qquad (F.4)$$

Using $(F.4)$, the Normal ordered product $(F.1)$ on the leptonic current is given by:

$$\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T\overline{e}_L^T(x_2)\right]\middle|0\right\rangle$$

$$= \frac{i}{2}\sum_k (U_{ek}^L)^2\, m_k\xi_k\langle p_2,s_2;p_1,s_1|N[\overline{e}(x_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{e}^T(x_2)]|0\rangle\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}. \quad (F.5)$$

Now we will calculate the following quantity using quantization technique of the Dirac fields for electrons:

$$\langle p_2,s_2;p_1,s_1|N[\overline{e}(x_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{e}^T(x_2)]|0\rangle. \qquad (F.6)$$

Since the general solutions of the Dirac equation is given by $(D.1.3)$, the free Dirac field for the electron, $e(x)$ can be written as:

$$e(x) = \sum_{s=\pm\frac{1}{2}}\int\frac{d^3\vec{p}}{(2\pi)^3 2E}\left[b(p,s)u(p,s)e^{-ipx} + d^\dagger(p,s)v(p,s)e^{ipx}\right], \qquad (F.7)$$

where $u(p,s)$ and $v(p,s)$ are the four- component free Dirac spinors for particles and for antiparticles with four-momentum, p and a definite spin projection, s respectively, given in $(B.1)$. Here, $E = (\vec{p}^2 + m_e^2)^{1/2}$, is the energy and $\vec{p}$ is the momentum of a Dirac electron of mass, m_e. $b^\dagger(p,s)$ and $b(p,s)$ represent the creation and annihilation operators, respectively for Dirac electron of given spin, s and four momentum, p, while the operators $d^\dagger(p,s)$ and $d(p,s)$ create and annihilate a positron of given spin, s and four momentum, p.

Since, $\bar{e}(x_1) = e^\dagger(x_1)\gamma^0$, we have

$$\overline{e}(x_1) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_1}\left[b^\dagger(p,s)\overline{u}(p,s)e^{ipx_1} + d(p,s)\overline{v}(p,s)e^{-ipx_1}\right], \qquad (F.8)$$

Similarly,

$$C\bar{e}^T(x_2) = \sum_{s'=\pm\frac{1}{2}} \int \frac{d^3\vec{p}'}{(2\pi)^3 2E_2}\left[b^\dagger(p',s')C\overline{u}^T(p',s')e^{ip'x_2} + d(p',s')C\overline{v}^T(p',s')e^{-ip'x_2}\right]. \quad (F.9)$$

In the free Dirac field, a state of one particle, $|p,s\rangle$, with four-momentum, p, and a definite spin projection, s, can be created by acing on the vacuum state, $|0\rangle$, with a creation operator, $b^\dagger(p,s)$:

$$|p,s\rangle = b^\dagger(p,s)|0\rangle, \qquad (F.10)$$

where, $\langle 0|0\rangle = 1$, and the Lorentz-invariant normalization of the one particle states are given by:

$$\langle p,s|p',s'\rangle = (2\pi)^3 2E\delta^3(\vec{p}-\vec{p}')\delta_{ss'}, \qquad E = (\vec{p}^2 + m_e^2)^{1/2} \qquad (F.11)$$

Similarly from $(F.10)$, two-particle state for the electrons with four-momentum, p_1, p_2 and spin projections, s_1, s_2 can be written as:

$$|p_2,s_2;p_1,s_1\rangle = b^\dagger(p_1,s_1)b^\dagger(p_2,s_2)|0\rangle. \qquad (F.12)$$

Thus the quantity in $(F.6)$ can be written as:

$$\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}(x_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{e}^T(x_2)\right]\middle|0\right\rangle$$

$$= \left\langle 0\middle|N\left[b(p_2,s_2)b(p_1,s_1)\overline{e}(x_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{e}^T(x_2)\right]\middle|0\right\rangle$$

$$= \sum_{s,s'} \int \int \frac{d^3\vec{p}}{(2\pi)^3 2E_1} \frac{d^3\vec{p}'}{(2\pi)^3 2E_2} e^{ipx_1 + ip'x_2} \bar{u}(p,s)\gamma^\mu(1-\gamma^5)\gamma^\nu C\bar{u}^T(p',s')$$

$$\times \langle 0 | b(p_2,s_2) b(p_1,s_1) b^\dagger(p,s) b^\dagger(p',s') | 0 \rangle. \quad (F.13)$$

In the last line we substitute the expressions from $(F.8)$ and $(F.9)$. Note that $(F.13)$ does not contain the operators $d^\dagger(p,s)$ and $d(p,s)$ which create and annihilate a positron, respectively, since in $0\nu\beta\beta$ decay process, at the final stage of decay, we have only two electrons and a daughter nuclei.

The quantity $\langle 0 | b(p_2,s_2) b(p_1,s_1) b^\dagger(p,s) b^\dagger(p',s') | 0 \rangle$ in $(F.13)$ can be computed using the anticommutation relations $(D.1.5)$ of the creation and annihilation operators for the free Dirac fields of electrons. Since from $(D.1.5)$, we have:

$$\{b(p_1,s_1), b^\dagger(p,s)\} = (2\pi)^3 2E_1 \delta^3(\vec{p}_1 - \vec{p}) \delta_{s_1 s},$$

the product of operators $b(p_1,s_1)b^\dagger(p,s)$ can be written as:

$$b(p_1,s_1) b^\dagger(p,s) = (2\pi)^3 2E_1 \delta^3(\vec{p}_1 - \vec{p}) \delta_{s_1 s} - b^\dagger(p,s) b(p_1,s_1). \quad (F.14)$$

Thus,

$$\langle 0 | b(p_2,s_2) b(p_1,s_1) b^\dagger(p,s) b^\dagger(p',s') | 0 \rangle$$

$$= (2\pi)^3 2E_1 \delta^3(\vec{p}_1 - \vec{p}) \delta_{s_1 s} \langle 0 | b(p_2,s_2) b^\dagger(p',s') | 0 \rangle$$

$$-\langle 0 | b(p_2,s_2) b^\dagger(p,s) b(p_1,s_1) b^\dagger(p',s') | 0 \rangle. \quad (F.15)$$

Similarly, from

$$b(p_2,s_2) b^\dagger(p',s') = (2\pi)^3 2E_2 \delta^3(\vec{p}_2 - \vec{p}') \delta_{s_2 s'} - b^\dagger(p',s') b(p_2,s_2),$$

we have,

$$\langle 0 | b(p_2,s_2) b^\dagger(p',s') | 0 \rangle = (2\pi)^3 2E_2 \delta^3(\vec{p}_2 - \vec{p}') \delta_{s_2 s'} \langle 0|0 \rangle - \langle 0 | b^\dagger(p',s') b(p_2,s_2) | 0 \rangle$$

$$= (2\pi)^3 2E_2 \delta^3(\vec{p}_2 - \vec{p}') \delta_{s_2 s'}\,, \quad (F.16)$$

where in the last line we used the fact that $\langle 0|0\rangle = 1$ and $b(p_2, s_2)|0\rangle = 0$. Again using the same technique one can write down:

$$\left\langle 0\middle|b(p_2,s_2)b^\dagger(p,s)b(p_1,s_1)b^\dagger(p',s')\middle|0\right\rangle$$

$$= (2\pi)^3 2E_1\delta^3(\vec{p}_1 - \vec{p}')\delta_{s_1s'}\left\langle 0\middle|b(p_2,s_2)b^\dagger(p,s)\middle|0\right\rangle$$

$$-\left\langle 0\middle|b(p_2,s_2)b^\dagger(p,s)b^\dagger(p',s')b(p_1,s_1)\middle|0\right\rangle$$

$$= (2\pi)^3 2E_1\delta^3(\vec{p}_1 - \vec{p}')\delta_{s_1s'}\left\langle 0\middle|b(p_2,s_2)b^\dagger(p,s)\middle|0\right\rangle$$

$$= (2\pi)^3 2E_1\delta^3(\vec{p}_1 - \vec{p}')\delta_{s_1s'}(2\pi)^3 2E_2\delta^3(\vec{p}_2 - \vec{p})\delta_{s_2s}\,. \qquad (F.17)$$

Substituting $(F.16)$ and $(F.17)$ into $(F.15)$, we have

$$\left\langle 0\middle|b(p_2,s_2)b(p_1,s_1)b^\dagger(p,s)b^\dagger(p',s')\middle|0\right\rangle$$

$$= (2\pi)^3 2E_1\delta^3(\vec{p}_1 - \vec{p})\delta_{s_1s}(2\pi)^3 2E_2\delta^3(\vec{p}_2 - \vec{p}')\delta_{s_2s'}$$

$$-(2\pi)^3 2E_1\delta^3(\vec{p}_1 - \vec{p}')\delta_{s_1s'}(2\pi)^3 2E_2\delta^3(\vec{p}_2 - \vec{p})\delta_{s_2s}\,. \qquad (F.18)$$

Substituting $(F.18)$ into $(F.13)$, summing over the spins s and s' and integrating over the momenta $\vec{p}$ and $\vec{p}'$, we get:

$$\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}(x_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{e}^T(x_2)\right]\middle|0\right\rangle$$

$$= e^{ip_1x_1+ip_2x_2}\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)$$

$$-e^{ip_2x_1+ip_1x_2}\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2). \qquad (F.19)$$

Substituting $(F.19)$ into $(F.5)$, we finally have our desired expression:

$$\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T\overline{e}_L^T(x_2)\right]\middle|0\right\rangle$$

$$= \frac{i}{2}\sum_k (U_{ek}^L)^2\, m_k\xi_k \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2 - m_k^2} \times$$

$$\Big[e^{ip_1x_1+ip_2x_2}\overline{u}(p_1,s_1)\gamma^{\mu}(1-\gamma^5)\gamma^{\nu}C\overline{u}^T(p_2,s_2)$$

$$-\,e^{ip_2x_1+ip_1x_2}\overline{u}(p_2,s_2)\gamma^{\mu}(1-\gamma^5)\gamma^{\nu}C\overline{u}^T(p_1,s_1)\Big]\,. \qquad (F.20)$$

APPENDIX G

CALCULATION OF THE PRODUCT OF NORMAL ORDERED PRODUCT OF THE LEPTONIC CURRENT AND THE TIME ORDERED PRODUCT OF THE HADRONIC (NUCLEAR) CURRENT

In this section we will calculate the product of Normal ordered product of the Leptonic current and the Time ordered product of the Hadronic (nuclear) current, that is, we will calculate the following quantity which appears as an integrand of the second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$ for $0\nu\beta\beta$ decay process:

$$\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T\overline{e}_L^T(x_2)\right]\middle|0\right\rangle\langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle, \quad (G.1)$$

where the Normal ordered product of the Leptonic current is calculated in the appendix F and the result is given by $(F.20)$. On the other hand, the quantity $\langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle$ is the time ordered product of the Hadronic (nuclear) current where $|p_i\rangle$ and $|p_f\rangle$ are the initial and final nuclear states in the Heisenberg representation with four-momenta p_i and p_f, respectively, $J_L^{\mu\dagger}(x)$ is the weak charged Hadronic current in the Heisenberg representation and T denotes the time ordering. The time ordered product of the Hadronic current is defined by:

$$\langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle = \begin{cases}\langle p_f|J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)|p_i\rangle & for \quad t_1>t_2 \\ \langle p_f|J_{\nu L}^\dagger(x_2)J_{\mu L}^\dagger(x_1)|p_i\rangle & for \quad t_2>t_1\end{cases}. \quad (G.2)$$

Substituting the expression for the normal product of leptonic current from $(F.20)$, we have:

$$\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T\overline{e}_L^T(x_2)\right]\middle|0\right\rangle\langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle$$

$$=\frac{i}{2}\sum_k (U_{ek}^L)^2\, m_k\xi_k\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}\langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle\times$$

$$\left[e^{ip_1x_1+ip_2x_2}\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)\right.$$

$$\left.-\,e^{ip_2x_1+ip_1x_2}\overline{u}(p_2,s_2)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_1,s_1)\right]. \qquad (G.3)$$

Now using $(G.3)$ the second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$ for $0\nu\beta\beta$ decay process can be written as:

$$\langle f|S^{(2)}|i\rangle^{0\nu}$$

$$= -2\left(\frac{G_F}{\sqrt{2}}\right)^2 \int\int d^4x_1 d^4x_2 \left\langle p_2,s_2;p_1,s_1\middle| N\left[\overline{e}_L(x_1)\gamma^\mu \overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}(\gamma^\nu)^T\overline{e}_L^T(x_2)\right]\middle|0\right\rangle$$

$$\times \langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle$$

$$= -i\left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U_{ek}^L)^2\, m_k\xi_k\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)\int\int d^4x_1d^4x_2\, e^{ip_1x_1+ip_2x_2}$$

$$\times \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}\langle p_f|T[J_{\mu L}^\dagger(x_1)J_{\nu L}^\dagger(x_2)]|p_i\rangle - (p_1,s_1 \rightleftarrows p_2,s_2), \qquad (G.4)$$

where $(p_1,s_1 \rightleftarrows p_2,s_2)$ denotes the fact that the second term is obtained by interchanging the four-momenta, p_1 and spin, s_1 with the four-momenta, p_2 and spin, s_2 of first term of $(G.4)$.

Now it can be proved that the first term in right hand side of the expression of $(G.4)$ is identical with the second term, and hence $(G.4)$ can be written as twice of the first term. This can be proved as follows:

$$\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2) = \left[\overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)\right]^T$$

$$= \overline{u}(p_2,s_2)C^T(\gamma^\nu)^T(1-\gamma^5)^T(\gamma^\mu)^T\overline{u}^T(p_1,s_1)$$

$$= -\overline{u}(p_2,s_2)C(\gamma^\nu)^TC^{-1}C(1-\gamma^5)^TC^{-1}C(\gamma^\mu)^TC^{-1}C\overline{u}^T(p_1,s_1)$$

$$= -\overline{u}(p_2,s_2)\gamma^\nu(1-\gamma^5)\gamma^\mu C\overline{u}^T(p_1,s_1). \qquad (G.5)$$

Here, we have used the fact that, as the quantity, $\overline{u}(p_1,s_1)\gamma^{\mu}(1-\gamma^5)\gamma^{\nu}C\overline{u}^T(p_2,s_2)$ is a 1×1 matrix, thus we can write it as its transpose and $C^T=-C,\ C(\gamma^{\mu})^TC^{-1}=-\gamma^{\mu}$, and $C(\gamma^5)^TC^{-1}=\gamma^5$.

Now with the identity $(G.5)$ as well as the possibility of interchanging the current operators under the sign of time ordered product, we can write:

$$\overline{u}(p_1,s_1)\gamma^{\mu}(1-\gamma^5)\gamma^{\nu}C\overline{u}^T(p_2,s_2)\int\int d^4x_1d^4x_2\,e^{ip_1x_1+ip_2x_2}\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}$$

$$\times\langle p_f|T[J_{\mu L}^{\dagger}(x_1)J_{\nu L}^{\dagger}(x_2)]|p_i\rangle$$

$$=-\overline{u}(p_2,s_2)\gamma^{\nu}(1-\gamma^5)\gamma^{\mu}C\overline{u}^T(p_1,s_1)\int\int d^4x_1d^4x_2\,e^{ip_2x_1+ip_1x_2}\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}$$

$$\times\langle p_f|T[J_{\mu L}^{\dagger}(x_1)J_{\nu L}^{\dagger}(x_2)]|p_i\rangle. \qquad (G.6)$$

Thus the second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$ for $0\nu\beta\beta$ decay process, given by $(G.4)$ can be written as:

$$\langle f|S^{(2)}|i\rangle^{0\nu}=-i\left(\frac{G_F}{\sqrt{2}}\right)^2\sum_k(U_{ek}^L)^2\,m_k\xi_k\overline{u}(p_1,s_1)\gamma^{\mu}(1-\gamma^5)\gamma^{\nu}C\overline{u}^T(p_2,s_2)$$

$$\times\int\int d^4x_1d^4x_2e^{ip_1x_1+ip_2x_2}\int\frac{d^4q}{\pi(2\pi)^3}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}\langle p_f|T[J_{\mu L}^{\dagger}(x_1)J_{\nu L}^{\dagger}(x_2)]|p_i\rangle. \qquad (G.7)$$

APPENDIX H

CALCULATION OF THE NEUTRINO MOMENTUM INTEGRAL

This section is devoted to the calculation of the neutrino momentum integral which appears in the expression of the second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$ for $0\nu\beta\beta$ decay process, given by $(G.7)$. We will calculate the following integral:

$$\int \frac{d^4q}{\pi(2\pi)^3}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2} \qquad (H.1)$$

where $q^\mu = (p-p')^\mu = (E,\vec{q})$ is the four-momentum transfer from hadrons to leptons, i.e., the four-momentum of the virtual neutrino with p and p' being the four-momenta of neutron and proton, respectively. E and $\vec{q}$ are the energy and momentum of the virtual neutrino. m_k's are the masses of the Majorana neutrino mass eigenstates.

Since,

$$q^2 - m_k^2 = E^2 - \vec{q}^2 - m_k^2 = E^2 - E_k^2, \qquad E_k = \sqrt{\vec{q}^2 - m_k^2}\,, \qquad (H.2)$$

$(H.1)$ can be written as:

$$\int \frac{d^4q}{\pi(2\pi)^3}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2} = \int_{-\infty}^{\infty}\frac{dE}{\pi}\frac{e^{-iE(t_1-t_2)}}{E^2-E_k^2}\int\frac{d^3\vec{q}}{(2\pi)^3}e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)} \qquad (H.3)$$

As the pole lies in the real axis of the complex plane at $E = \pm E_k$, the integral,

$$\int_{-\infty}^{\infty}\frac{dE}{\pi}\frac{e^{-iE(t_1-t_2)}}{E^2-E_k^2}, \qquad (H.4)$$

diverges, thus we compute the integral $(H.4)$ shifting the pole in the complex plane at $E = \pm E_k \mp i\gamma$ where $\gamma \to 0$, which is called the principal value of the integral $(H.4)$, denoted by $\mathcal{P}$.

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{dE}{\pi}\frac{e^{-iE(t_1-t_2)}}{E^2-E_k^2} = \lim_{\gamma\to0}\int_{-\infty}^{\infty}\frac{dE}{2\pi E_k}\frac{e^{-iE(t_1-t_2)}}{E-E_k+i\gamma} - \lim_{\gamma\to0}\int_{-\infty}^{\infty}\frac{dE}{2\pi E_k}\frac{e^{-iE(t_1-t_2)}}{E+E_k-i\gamma}$$

$$= I_1 + I_2 \,, \tag{H.5}$$

where

$$I_1 = \lim_{\gamma \to 0} \int_{-\infty}^{\infty} \frac{dE}{2\pi E_k} \frac{e^{-iE(t_1 - t_2)}}{E - E_k + i\gamma}, \qquad I_2 = -\lim_{\gamma \to 0} \int_{-\infty}^{\infty} \frac{dE}{2\pi E_k} \frac{e^{-iE(t_1 - t_2)}}{E + E_k - i\gamma}. \tag{H.6}$$

Applying residue theorem we can compute the contour integral $(H.5)$. We have the following two cases:

Case 1: $(t_1 > t_2)$

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{dE}{\pi} \frac{e^{-iE(t_1 - t_2)}}{q^2 - m_k^2} = I_1 = \frac{-2\pi i e^{-iE_k(t_1 - t_2)}}{2\pi E_k} = -\frac{i}{E_k} e^{-iE_k(t_1 - t_2)}. \tag{H.7}$$

Case 2: $(t_2 > t_1)$

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{dE}{\pi} \frac{e^{-iE(t_1 - t_2)}}{q^2 - m_k^2} = I_2 = -\frac{2\pi i e^{iE_k(t_1 - t_2)}}{2\pi E_k} = -\frac{i}{E_k} e^{iE_k(t_1 - t_2)}. \tag{H.8}$$

Thus the integral $(H.3)$ can be written as:

$$\int \frac{d^4 q}{\pi (2\pi)^3} \frac{e^{-iq(x_1 - x_2)}}{q^2 - m_k^2} = \begin{cases} -i \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{e^{-iE_k(t_1 - t_2)} e^{i\vec{q}.(\vec{x}_1 - \vec{x}_2)}}{E_k} & for \qquad t_1 > t_2 \\ -i \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{e^{iE_k(t_1 - t_2)} e^{i\vec{q}.(\vec{x}_1 - \vec{x}_2)}}{E_k} & for \qquad t_2 > t_1 \end{cases} . \tag{H.9}$$

APPENDIX I

CALCULATION OF SECOND-ORDER PERTURBATIVE MATRIX ELEMENTS FOR THE $0\nu\beta\beta$ DECAY PROCESS

In this section we will calculate the second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$ for $0\nu\beta\beta$ decay process, given in $(G.7)$. In order to calculate the time ordered product of the Hadronic current, i.e., in order to compute the quantity, $\langle p_f|T[J^{\dagger}_{\mu L}(x_1)J^{\dagger}_{\nu L}(x_2)]|p_i\rangle$, which appears in the expression for $\langle f|S^{(2)}|i\rangle^{0\nu}$, given in $(G.7)$, we split the four-vector current, $J_L^{\mu\dagger}(x)$ into space and time components inserting the completeness relation for the nuclear states.

Thus using the completeness relation for the nuclear states, $|n\rangle$:

$$\sum_n |n\rangle\langle n| = 1, \tag{I.1}$$

we can write,

$$\langle p_f|J^{\dagger}_{\mu L}(x_1)J^{\dagger}_{\nu L}(x_2)|p_i\rangle = \sum_n \langle p_f|J^{\dagger}_{\mu L}(\vec{x}_1)|n\rangle\langle n|J^{\dagger}_{\nu L}(\vec{x}_2)|p_i\rangle\, e^{i(E_f-E_n)t_1}e^{i(E_n-E_i)t_2},$$

$$\langle p_f|J^{\dagger}_{\nu L}(x_2)J^{\dagger}_{\mu L}(x_1)|p_i\rangle = \sum_n \langle p_f|J^{\dagger}_{\nu L}(\vec{x}_2)|n\rangle\langle n|J^{\dagger}_{\mu L}(\vec{x}_1)|p_i\rangle\, e^{i(E_f-E_n)t_2}e^{i(E_n-E_i)t_1}, \tag{I.2}$$

where E_i, E_f and E_n are the energies of the initial, final and intermediate nuclear states, respectively.

Since, the time ordered product of the Hadronic current is defined by:

$$\langle p_f|T[J^{\dagger}_{\mu L}(x_1)J^{\dagger}_{\nu L}(x_2)]|p_i\rangle = \begin{cases}\langle p_f|J^{\dagger}_{\mu L}(x_1)J^{\dagger}_{\nu L}(x_2)|p_i\rangle & for \quad t_1 > t_2 \\ \langle p_f|J^{\dagger}_{\nu L}(x_2)J^{\dagger}_{\mu L}(x_1)|p_i\rangle & for \quad t_2 > t_1\end{cases}, \tag{G.2}$$

and the neutrino momentum integral is given by:

$$\int \frac{d^4q}{\pi(2\pi)^3}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2} = \begin{cases} -i\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{-iE_k(t_1-t_2)}e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k} & for \quad t_1 > t_2 \\ -i\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{iE_k(t_1-t_2)}e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k} & for \quad t_2 > t_1 \end{cases}, \quad (H.9)$$

we can compute the following quantity present in the expression for $\langle f|S^{(2)}|i\rangle^{0\nu}$:

$$\int\int d^4x_1 d^4x_2 e^{ip_1x_1+ip_2x_2}\int \frac{d^4q}{\pi(2\pi)^3}\frac{e^{-iq(x_1-x_2)}}{q^2-m_k^2}\langle p_f|T[J^\dagger_{\mu L}(x_1)J^\dagger_{\nu L}(x_2)]|p_i\rangle$$

$$= -i\int\int d^3\vec{x}_1 d^3\vec{x}_2 e^{-i\vec{p}_1.\vec{x}_1-i\vec{p}_2.\vec{x}_2}\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k}$$

$$\times\begin{cases} \int\int dt_1dt_2 e^{iE_1t_1+iE_2t_2}e^{-iE_k(t_1-t_2)}\langle p_f|J^\dagger_{\mu L}(x_1)J^\dagger_{\nu L}(x_2)|p_i\rangle & for \quad t_1 > t_2 \\ \int\int dt_1dt_2 e^{iE_1t_1+iE_2t_2}e^{iE_k(t_1-t_2)}\langle p_f|J^\dagger_{\nu L}(x_2)J^\dagger_{\mu L}(x_1)|p_i\rangle & for \quad t_2 > t_1 \end{cases} \quad (I.3)$$

where p_1 and p_2 represent the four-momenta of electrons, E_1 and E_2 are the energies and $\vec{p}_1$ and $\vec{p}_2$ are the momenta of electrons.

The time ordered integrals in $(I.3)$ can be calculated by substituting the equations $(I.2)$. The case for which $t_1 > t_2$, we have:

$$\int\int dt_1dt_2 e^{iE_1t_1+iE_2t_2}e^{-iE_k(t_1-t_2)}\langle p_f|J^\dagger_{\mu L}(x_1)J^\dagger_{\nu L}(x_2)|p_i\rangle$$

$$= \sum_n \langle p_f|J^\dagger_{\mu L}(\vec{x}_1)|n\rangle\langle n|J^\dagger_{\nu L}(\vec{x}_2)|p_i\rangle\int_{-\infty}^{\infty} dt_1\, e^{i(E_f-E_k+E_1-E_n)t_1}\int_{-\infty}^{t_1} dt_2\, e^{i(E_n+E_k+E_2-E_i)t_2}$$

$$= \sum_n \frac{\langle p_f|J^\dagger_{\mu L}(\vec{x}_1)|n\rangle\langle n|J^\dagger_{\nu L}(\vec{x}_2)|p_i\rangle}{i(E_n+E_k+E_2-E_i)}\int_{-\infty}^{\infty} dt_1\, e^{i(E_f-E_k+E_1-E_n)t_1}e^{i(E_n+E_k+E_2-E_i)t_1}$$

$$= \sum_n \frac{\langle p_f|J^\dagger_{\mu L}(\vec{x}_1)|n\rangle\langle n|J^\dagger_{\nu L}(\vec{x}_2)|p_i\rangle}{i(E_n+E_k+E_2-E_i)}\int_{-\infty}^{\infty} dt_1\, e^{i(E_f+E_1+E_2-E_i)t_1}$$

$$= \sum_n \frac{\langle p_f|J^\dagger_{\mu L}(\vec{x}_1)|n\rangle\langle n|J^\dagger_{\nu L}(\vec{x}_2)|p_i\rangle}{i(E_n+E_k+E_2-E_i)}2\pi\delta(E_f+E_1+E_2-E_i). \quad (I.4)$$

Note that in the second line the integrals appears according to the ordering of time, i.e., according to $t_1 > t_2$. The integral in the third line is evaluated using standard adiabatic switch-off procedure. (see, for example, the derivation of Fermi's Golden Rule in appendix O.1).

Similarly, for the case $t_2 > t_1$, we have:

$$\int\int dt_1 dt_2 e^{iE_1t_1+iE_2t_2} e^{iE_k(t_1-t_2)} \langle p_f | J_{\nu L}^\dagger(x_2) J_{\mu L}^\dagger(x_1) | p_i \rangle$$

$$= \sum_n \frac{\langle p_f | J_{\nu L}^\dagger(\vec{x}_2) | n \rangle \langle n | J_{\mu L}^\dagger(\vec{x}_1) | p_i \rangle}{i(E_n + E_k + E_1 - E_i)} 2\pi\delta(E_f + E_1 + E_2 - E_i). \qquad (I.5)$$

Since $0\nu\beta\beta$ decay includes both the possibility $t_1 > t_2$ and $t_2 > t_1$, we have to add the results from $(I.4)$ and $(I.5)$. Thus $(I.3)$ can be written as:

$$\int\int d^4x_1 d^4x_2 e^{ip_1x_1+ip_2x_2} \int \frac{d^4q}{\pi(2\pi)^3} \frac{e^{-iq(x_1-x_2)}}{q^2 - m_k^2} \langle p_f | T[J_{\mu L}^\dagger(x_1) J_{\nu L}^\dagger(x_2)] | p_i \rangle$$

$$= -2\pi\delta(E_f + E_1 + E_2 - E_i) \int\int d^3\vec{x}_1 d^3\vec{x}_2 e^{-i\vec{p}_1.\vec{x}_1 - i\vec{p}_2.\vec{x}_2} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k}$$

$$\times \sum_n \left[\frac{\langle p_f | J_{\mu L}^\dagger(\vec{x}_1) | n \rangle \langle n | J_{\nu L}^\dagger(\vec{x}_2) | p_i \rangle}{E_n + E_k + E_2 - E_i} + \frac{\langle p_f | J_{\nu L}^\dagger(\vec{x}_2) | n \rangle \langle n | J_{\mu L}^\dagger(\vec{x}_1) | p_i \rangle}{E_n + E_k + E_1 - E_i} \right]. \qquad (I.6)$$

Substituting $(I.6)$ into $(G.7)$, we have our expression for the second order perturbative matrix element, $\langle f | S^{(2)} | i \rangle^{0\nu}$ for $0\nu\beta\beta$ decay process:

$$\langle f | S^{(2)} | i \rangle^{0\nu} = i \left(\frac{G_F}{\sqrt{2}} \right)^2 \sum_k (U_{ek}^L)^2 m_k \xi_k \overline{u}(p_1, s_1) \gamma^\mu (1 - \gamma^5) \gamma^\nu C \overline{u}^T(p_2, s_2)$$

$$\times \int\int d^3\vec{x}_1 d^3\vec{x}_2 e^{-i\vec{p}_1.\vec{x}_1 - i\vec{p}_2.\vec{x}_2} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k} \times$$

$$\sum_n \left[\frac{\langle p_f | J_{\mu L}^\dagger(\vec{x}_1) | n \rangle \langle n | J_{\nu L}^\dagger(\vec{x}_2) | p_i \rangle}{E_n + E_k + E_2 - E_i} + \frac{\langle p_f | J_{\nu L}^\dagger(\vec{x}_2) | n \rangle \langle n | J_{\mu L}^\dagger(\vec{x}_1) | p_i \rangle}{E_n + E_k + E_1 - E_i} \right] 2\pi\delta(E_f + E_1 + E_2 - E_i).$$

$$\dots\dots..(I.7)$$

Note in $(I.7)$, the Dirac delta function represents the conservation of energy, i.e., $E_i = E_f + E_1 + E_2$.

The following two approximation will be made for the second order perturbative matrix element in $(I.7)$:

(i) Closure approximation: The energies E_n of the intermediate states $|n\rangle$ are replaced by the average energy $E_n \to \langle E_n \rangle = E_j$. This allows us to perform the summation over the complete system of states $|n\rangle$ using the completeness relation $(I.1)$ for nuclear states.

(ii) Long wave approximations for leptonic waves: Since the de Broglie wavelengths associated with the momenta $\vec{p}_1$ and $\vec{p}_2$ of electrons are very small compared to the nuclear radius R_A of the decaying nuclei, i.e. $|\vec{p}_{1,2}|R_A \ll 1$, we can assume $e^{-i\vec{p}_1.\vec{x}_1 - i\vec{p}_2.\vec{x}_2} \to 1$.

With the above approximations, we can write $(I.7)$ as:

$$\langle f|S^{(2)}|i\rangle^{0\nu} = i\left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U_{ek}^L)^2\, m_k \xi_k \overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)$$

$$\times \left\langle \psi_f \middle| \int\int d^3\vec{x}_1 d^3\vec{x}_2 \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k}\left[\frac{1}{E_k+E_j+E_1-E_i} + \frac{1}{E_k+E_j+E_2-E_i}\right] J_{\mu L}^\dagger(\vec{x}_1) J_{\nu L}^\dagger(\vec{x}_2) \middle| \psi_i \right\rangle$$

$$\times 2\pi\delta\left(E_f + E_1 + E_2 - E_i\right), \qquad (I.8)$$

where ψ_i and ψ_f are the wave functions of the initial and final nuclear state.

Note $(I.8)$ can be written down as:

$$\langle f|S^{(2)}|i\rangle^{0\nu} = \overline{u}(p_1,s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2,s_2)A_{\mu\nu}, \qquad (I.9)$$

with

$$A_{\mu\nu} = i\left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U_{ek}^L)^2\, m_k \xi_k$$

$$\times \left\langle \psi_f \middle| \int \int d^3\vec{x}_1 d^3\vec{x}_2 \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k} \left[\frac{1}{E_k + E_j + E_1 - E_i} + \frac{1}{E_k + E_j + E_2 - E_i}\right] J^{\dagger}_{\mu L}(\vec{x}_1) J^{\dagger}_{\nu L}(\vec{x}_2) \middle| \psi_i \right\rangle$$

$$\times 2\pi\delta\left(E_f + E_1 + E_2 - E_i\right). \qquad (I.10)$$

Using the anti-commutation relations of the Dirac gamma matrices, given by $(A.2.2)$, we can write down:

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} + \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \qquad (I.11)$$

where, $g^{\mu\nu} = diag(1, -1, -1, -1)$, is the metric tensor. Thus $(I.9)$ becomes:

$$\langle f | S^{(2)} | i \rangle^{0\nu} = \overline{u}(p_1, s_1)\gamma^\mu(1-\gamma^5)\gamma^\nu C\overline{u}^T(p_2, s_2)A_{\mu\nu} = \overline{u}(p_1, s_1)\gamma^\mu\gamma^\nu(1+\gamma^5)C\overline{u}^T(p_2, s_2)A_{\mu\nu}$$

$$= \overline{u}(p_1, s_1)g^{\mu\nu}(1+\gamma^5)C\overline{u}^T(p_2, s_2)A_{\mu\nu} + \frac{1}{2}\overline{u}(p_1, s_1)\gamma^\mu\gamma^\nu(1+\gamma^5)C\overline{u}^T(p_2, s_2)A_{\mu\nu}$$

$$-\frac{1}{2}\overline{u}(p_1, s_1)\gamma^\nu\gamma^\mu(1+\gamma^5)C\overline{u}^T(p_2, s_2)A_{\mu\nu} = \overline{u}(p_1, s_1)g^{\mu\nu}(1+\gamma^5)C\overline{u}^T(p_2, s_2)A_{\mu\nu}. \qquad (I.12)$$

Substituting $A_{\mu\nu}$ from $(I.10)$, into $(I.12)$, we finally have our expression for the second order perturbative matrix element, $\langle f | S^{(2)} | i \rangle^{0\nu}$ for $0\nu\beta\beta$ decay process:

$$\langle f | S^{(2)} | i \rangle^{0\nu} = i\left(\frac{G_F}{\sqrt{2}}\right)^2 \sum_k (U^L_{ek})^2\, m_k \xi_k \overline{u}(p_1, s_1)(1+\gamma^5)C\overline{u}^T(p_2, s_2)$$

$$\times \left\langle \psi_f \middle| \int \int d^3\vec{x}_1 d^3\vec{x}_2 \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{E_k} \left[\frac{1}{E_k + E_j + E_1 - E_i} + \frac{1}{E_k + E_j + E_2 - E_i}\right] J^{\dagger}_{\mu L}(\vec{x}_1) J^{\mu\dagger}_{L}(\vec{x}_2) \middle| \psi_i \right\rangle$$

$$\times 2\pi\delta\left(E_f + E_1 + E_2 - E_i\right). \qquad (I.13)$$

APPENDIX J

BILINEAR FORMS, BREIT FRAME AND SPINOR MATRIX ELEMENT IN THE LOW ENERGY TRANSFER (NON-RELATIVISTIC) APPROXIMATION

J.1. Bilinear forms and equivalent two-component free spinor matrix elements:

There are five bilinear combinations of Γ-matrices, $\Gamma = \mathbb{1},\ \gamma^5,\ \gamma^\mu,\ \gamma^\mu\gamma^5,\ \sigma^{\mu\nu}$ and Dirac field, $\psi(x)$:

$S = \overline{\psi}(x)\psi(x) = scaler\ (one\ component)$,

$P = \overline{\psi}(x)\gamma^5\psi(x) = pseudoscaler\ (one\ component)$,

$V^\mu = \overline{\psi}(x)\gamma^\mu\psi(x) = vector\ (four\ components)$,

$A^\mu = \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x) = pseudovector\ (four\ components)$,

$T^{\mu\nu} = \overline{\psi}(x)\sigma^{\mu\nu}\psi(x) = antisymmetric\ tensor\ (six\ components)$,

where, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$.

Using free positive energy Dirac spinors, given in $(B.1)$,

$$u(p,s) = \left(\frac{E+m}{2m}\right)^{\frac{1}{2}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\chi_s \end{pmatrix}, \qquad (B.1)$$

the Dirac tensor matrix elements $M(p',p)$ are given by:

$$\bar{u}(p',s')\Gamma u(p,s) = \chi_{s'}^{\dagger} M(p',p)\chi_s, \qquad (J.1.1)$$

where, $M(p',p)$ have values according to table 5. Here, E is the energy, p is the four-momentum of a Dirac Particle of mass, m. χ_s ‘s are a two-component Pauli spinors with $\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. s denotes the spin projection.

Equivalent two-component free spinor matrix elements [93].

$Type$	Γ	$M(p',p)$
S	$\mathbb{1}$	$N'N\left[1-\frac{(\vec{\sigma}.\vec{p}')(\vec{\sigma}.\vec{p})}{(E'+m)(E+m)}\right]$
P	γ^5	$N'N\left[\frac{\vec{\sigma}.\vec{p}}{E+m}-\frac{\vec{\sigma}.\vec{p}'}{E'+m}\right]$
V^0	γ^0	$N'N\left[1+\frac{(\vec{\sigma}.\vec{p}')(\vec{\sigma}.\vec{p})}{(E'+m)(E+m)}\right]$
$\vec{V}$	$\vec{\gamma}$	$N'N\left[\vec{\sigma}\frac{\vec{\sigma}.\vec{p}}{E+m}+\frac{\vec{\sigma}.\vec{p}'}{E'+m}\vec{\sigma}\right]$
A^0	$\gamma^0\gamma^5$	$N'N\left[\frac{\vec{\sigma}.\vec{p}'}{E'+m}+\frac{\vec{\sigma}.\vec{p}}{E+m}\right]$
$\vec{A}$	$\vec{\gamma}\gamma^5$	$N'N\left[\vec{\sigma}+\frac{\vec{\sigma}.\vec{p}'}{E'+m}\vec{\sigma}\frac{\vec{\sigma}.\vec{p}}{E+m}\right]$
T^{0j}	σ^{0j}	$N'Ni\left[\sigma^j\frac{\vec{\sigma}.\vec{p}}{E+m}-\frac{\vec{\sigma}.\vec{p}'}{E'+m}\sigma^j\right]$
T^{ij}	σ^{ij}	$N'N\left[\sigma^k-\frac{\vec{\sigma}.\vec{p}'}{E'+m}\sigma^k\frac{\vec{\sigma}.\vec{p}}{E+m}\right]\varepsilon_{ijk}$

Here the normalization constants are: $N=\left(\frac{E+m}{2m}\right)^{1/2}$ and $N'=\left(\frac{E'+m}{2m}\right)^{1/2}$.

J.2. Non-Relativistic expansion of two-component free spinor matrix elements:

The expressions for two-component free spinor matrix elements, $M(p',p)$, in table 6 are obtained by expanding to the order of ω/m in the energy transfer, $\omega=E'-E$, assuming

$\omega/m \ll 1$. Here E, E' and $\bar{E}$ are the initial, final and average energies and $\vec{p}$ and $\vec{p}'$ are initial and final momenta of a Dirac particle of mass, m. E, E' and $\bar{E}$ are given by:

$$E = (\vec{p}^2 + m^2)^{\frac{1}{2}}, \qquad E' = (\vec{p}'^2 + m^2)^{\frac{1}{2}}, \qquad \bar{E} = (E + E')/2. \qquad (J.2.1)$$

Equivalent two-component free spinor matrix elements for low energy transfers ($\omega/m \ll 1$) [93].

$Type$	Γ	$M(p', p)$
S	$\mathbb{1}$	$\frac{\bar{E}+m}{2m}\left[1 - \frac{(\vec{\sigma}.\vec{p}')(\vec{\sigma}.\vec{p})}{(\bar{E}+m)^2}\right] + O\left(\frac{\omega^2}{m^2}\right)$
P	γ^5	$-\frac{\vec{\sigma}.(\vec{p}'-\vec{p})}{2m} + \frac{\omega}{4m}\frac{\vec{\sigma}.(\vec{p}'+\vec{p})}{\bar{E}+m} + O\left(\frac{\omega^2}{m^2}\right)$
V^0	γ^0	$\frac{\bar{E}+m}{2m}\left[1 + \frac{(\vec{\sigma}.\vec{p}')(\vec{\sigma}.\vec{p})}{(\bar{E}+m)^2}\right] + O\left(\frac{\omega^2}{m^2}\right)$
$\vec{V}$	$\vec{\gamma}$	$\frac{\vec{p}+\vec{p}'+i\vec{\sigma}\times(\vec{p}'-\vec{p})}{2m} + \frac{\omega}{4m}\frac{\vec{p}-\vec{p}'-i\vec{\sigma}\times(\vec{p}+\vec{p}')}{\bar{E}+m} + O\left(\frac{\omega^2}{m^2}\right)$
A^0	$\gamma^0\gamma^5$	$\frac{\vec{\sigma}.(\vec{p}+\vec{p}')}{2m} - \frac{\omega}{4m}\frac{\vec{\sigma}.(\vec{p}'-\vec{p})}{\bar{E}+m} + O\left(\frac{\omega^2}{m^2}\right)$
$\vec{A}$	$\vec{\gamma}\gamma^5$	$\frac{\bar{E}+m}{2m}\left[\vec{\sigma} + \frac{\vec{\sigma}.\vec{p}'}{\bar{E}+m}\vec{\sigma}\frac{\vec{\sigma}.\vec{p}}{\bar{E}+m}\right] + O\left(\frac{\omega^2}{m^2}\right)$
T^{0j}	σ^{0j}	$i\frac{[\vec{p}-\vec{p}']^j}{2m} + \frac{[\vec{\sigma}\times(\vec{p}+\vec{p}')]^j}{2m} + \frac{\omega}{4m}\left[i\frac{[\vec{p}+\vec{p}]^j}{\bar{E}+m} + \frac{[\vec{\sigma}\times(\vec{p}-\vec{p}')]^j}{\bar{E}+m}\right] + O\left(\frac{\omega^2}{m^2}\right)$
T^{ij}	σ^{ij}	$\frac{\bar{E}+m}{2m}\left[\sigma^k - \frac{\vec{\sigma}.\vec{p}'}{\bar{E}+m}\sigma^k\frac{\vec{\sigma}.\vec{p}}{\bar{E}+m}\right]\varepsilon_{ijk} + O\left(\frac{\omega^2}{m^2}\right)$

J.3. Expansion of two-component free spinor matrix elements in the Breit frame:

Breit frame is defined by, $\vec{p} = -\vec{p}\,' = \vec{q}/2$. It follows that in the Breit frame, $\vec{p} + \vec{p}\,' = \mathbf{0}, \bar{E} = E' = E = (\vec{q}^{\,2}/4 + m^2)^{\frac{1}{2}}$ and $\omega = E' - E = 0$. The two-component free spinor matrix elements, $M(p', p)$, in the Breit frame are given in the following table.

Non-relativistic expansion of two-component free spinor matrix elements in the Breit frame

$Type$	Γ	$M(p',p)$
S	$\mathbb{1}$	$\frac{E+m}{2m}\left[1+\frac{\vec{q}^{\,2}}{2(E+m)^2}\right] \approx \left[1+\frac{\vec{q}^{\,2}}{2(2m)^2}\right] \approx 1$
P	γ^5	$\frac{\vec{\sigma}.\vec{q}}{2m}$
V^0	γ^0	$\frac{E+m}{2m}\left[1-\frac{\vec{q}^{\,2}}{2(E+m)^2}\right] \approx \left[1-\frac{\vec{q}^{\,2}}{2(2m)^2}\right] \approx 1$
$\vec{V}$	$\vec{\gamma}$	$-i\frac{\vec{\sigma}\times\vec{q}}{2m}$
A^0	$\gamma^0\gamma^5$	0
$\vec{A}$	$\vec{\gamma}\gamma^5$	$\frac{E+m}{2m}\left[\vec{\sigma}-\frac{(\vec{\sigma}.\vec{q})\vec{\sigma}(\vec{\sigma}.\vec{q})}{4(E+m)^2}\right] \approx \left[\vec{\sigma}-\frac{(\vec{\sigma}.\vec{q})\vec{\sigma}(\vec{\sigma}.\vec{q})}{4(2m)^2}\right] \approx \vec{\sigma}$
T^{0j}	σ^{0j}	$i\frac{q^j}{2m}$
T^{ij}	σ^{ij}	$\frac{E+m}{2m}\left[\sigma^k+\frac{(\vec{\sigma}.\vec{q})\sigma^k(\vec{\sigma}.\vec{q})}{4(E+m)^2}\right]\varepsilon_{ijk} \approx \left[\sigma^k+\frac{(\vec{\sigma}.\vec{q})\sigma^k(\vec{\sigma}.\vec{q})}{4(2m)^2}\right]\varepsilon_{ijk}$ $\approx \sigma^k\varepsilon_{ijk}$

APPENDIX K

CALCULATION OF THE HIGHER ORDER TERMS IN THE HADRONIC CURRENT

In this section we will show that Gamow-Teller, $M_{GT}^{0\nu}$, Fermi, $M_F^{0\nu}$, and Tensor, $M_T^{0\nu}$, nuclear matrix elements given by (5.4.21) can be written in the form given by (5.4.22). In order to show this, we need to evaluate the following two integrals:

$$I_1 = \int d^3\vec{q}\, e^{i\vec{q}.\vec{x}}\,, \qquad I_2 = \int d^3\vec{q}\, q_i q_j e^{i\vec{q}.\vec{x}}, \qquad i,j = 1,2,3. \tag{K.1}$$

Now,

$$I_1 = \int d^3\vec{q}\, e^{i\vec{q}.\vec{x}} = 2\pi \int e^{i\mathrm{qx}cos\theta}\, \mathrm{q}^2 d\mathrm{q} sin\theta d\theta = 2\pi \int_0^\infty \left[\int_0^\pi e^{i\mathrm{qx}cos\theta} sin\theta d\theta\right] \mathrm{q}^2 d\mathrm{q}, \tag{K.2}$$

where, $d^3\vec{q} = 2\pi \mathrm{q}^2 d\mathrm{q} sin\theta d\theta$, $\mathrm{q} = |\vec{q}|$, $\mathrm{x} = |\vec{x}|$, and θ is the angle between the vectors, $\vec{q}$ and $\vec{x}$.

Making change of variable, $u = cos\theta$, we can evaluate the θ-integral:

$$\int_0^\pi e^{i\mathrm{qx}cos\theta} sin\theta d\theta = \int_{-1}^1 e^{i\mathrm{qx}u} du = \frac{2sin(\mathrm{qx})}{\mathrm{qx}} = 2j_0(\mathrm{qx}), \tag{K.3}$$

where, $j_0(\mathrm{qx}) = sin(\mathrm{qx})/\mathrm{qx}$, is the spherical Bessel function. Substituting the integral $(K.3)$ into the integral equation $(K.2)$, we get:

$$I_1 = \int d^3\vec{q}\, e^{i\vec{q}.\vec{x}} = 4\pi \int_0^\infty j_0(\mathrm{qx}) \mathrm{q}^2 d\mathrm{q}. \tag{K.4}$$

To evaluate the I_2-integral, note that

$$q_i q_j e^{i\vec{q}.\vec{x}} = -\frac{\partial^2}{\partial x_i \partial x_j}\left(e^{i\vec{q}.\vec{x}}\right), \qquad i,j = 1,2,3, \tag{K.5}$$

thus from $(K.4)$ and $(K.5)$, we have

$$I_2 = \int d^3\vec{q}\, q_i q_j e^{i\vec{q}.\vec{x}} = -\frac{\partial^2}{\partial x_i \partial x_j} \int d^3\vec{q}\, e^{i\vec{q}.\vec{x}} = -4\pi \int_0^\infty \frac{\partial^2}{\partial x_i \partial x_j}\left(j_0(\mathrm{qx})\right) \mathrm{q}^2 d\mathrm{q}. \tag{K.6}$$

Now,

$$\frac{\partial}{\partial x_j} \equiv \frac{\partial(\mathrm{qx})}{\partial x_j}\frac{\partial}{\partial(\mathrm{qx})} \equiv \frac{\mathrm{q}x_j}{\mathrm{x}}\frac{\partial}{\partial(\mathrm{qx})}, \qquad (K.7)$$

and

$$\frac{\partial^2}{\partial x_i \partial x_j} \equiv \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\right) \equiv \mathrm{q}\left(\frac{\delta_{ij}}{\mathrm{x}} - \frac{x_i x_j}{\mathrm{x}^3}\right)\frac{\partial}{\partial(\mathrm{qx})} + \frac{\mathrm{q}^2 x_i x_j}{\mathrm{x}^2}\frac{\partial^2}{\partial(\mathrm{qx})^2}, \qquad (K.8)$$

thus, $(K.6)$ becomes

$$I_2 = \int d^3\vec{q}\, q_i q_j e^{i\vec{q}.\vec{x}} = -4\pi\left[\int_0^\infty \left(\frac{\delta_{ij}}{\mathrm{x}} - \frac{x_i x_j}{\mathrm{x}^3}\right) j_0'(\mathrm{qx})\mathrm{q}^3 d\mathrm{q} + \int_0^\infty \frac{x_i x_j}{\mathrm{x}^2} j_0''(\mathrm{qx})\mathrm{q}^4 d\mathrm{q}\right]. \qquad (K.9)$$

Using the results for the I_1-integral, derived in $(K.4)$, the Gamow-Teller, $M_{GT}^{0\nu}$ and Fermi, $M_F^{0\nu}$, nuclear matrix elements given by (5.4.21), can be written as:

$$M_{GT}^{0\nu} = \frac{R_A}{2\pi^2}\left\langle\psi_f\left|\sum_{n,m}\int d^3\vec{q}\,\frac{e^{i\vec{q}.(\vec{x}_n - \vec{x}_m)}}{\mathrm{q}}\frac{1}{\mathrm{q} + E_j - (E_i + E_f)/2}\tau_n^-\tau_m^-\, h_{GT}(q^2)\vec{\sigma}_n.\vec{\sigma}_m\right|\psi_i\right\rangle,$$

$$= \left\langle\psi_f\left|\sum_{n,m}\tau_n^-\tau_m^-(\vec{\sigma}_n.\vec{\sigma}_m)\frac{2R_A}{\pi}\int_0^\infty \frac{j_0(\mathrm{q}x_{nm})h_{GT}(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2}\right|\psi_i\right\rangle, \qquad (K.10)$$

and

$$M_F^{0\nu} = \frac{R_A}{2\pi^2}\left\langle\psi_f\left|\sum_{n,m}\int d^3\vec{q}\,\frac{e^{i\vec{q}.(\vec{x}_n - \vec{x}_m)}}{\mathrm{q}}\frac{1}{\mathrm{q} + E_j - (E_i + E_f)/2}\tau_n^-\tau_m^-\, h_F(q^2)\right|\psi_i\right\rangle,$$

$$= \left\langle\psi_f\left|\sum_{n,m}\tau_n^-\tau_m^-\frac{2R_A}{\pi}\int_0^\infty \frac{j_0(\mathrm{q}x_{nm})h_F(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2}\right|\psi_i\right\rangle, \qquad (K.11)$$

where, $x_{nm} = |\vec{x}_n - \vec{x}_m|$.

To calculate the tensor nuclear matrix elements, $M_T^{0\nu}$ given by (5.4.21), note that the operator, S_{nm}^q given in (5.4.11), can be written as:

$$S_{nm}^q = 3(\vec{\sigma}_n.\hat{q})(\vec{\sigma}_m.\hat{q}) - \vec{\sigma}_n.\vec{\sigma}_m = \frac{3(\vec{\sigma}_n.\vec{q})(\vec{\sigma}_m.\vec{q})}{\mathrm{q}^2} - \vec{\sigma}_n.\vec{\sigma}_m$$

$$= 3\left(\sum_{i,j}\sigma_{ni}\sigma_{mj}q_i q_j\right)\Big/\mathrm{q}^2 - \vec{\sigma}_n.\vec{\sigma}_m, \qquad i,j = 1,2,3. \qquad (K.12)$$

Substituting $(K.12)$ into the expression for $M_T^{0\nu}$, we have:

$$M_T^{0\nu} = -\frac{R_A}{2\pi^2}\left\langle\psi_f\middle|\sum\nolimits_{n,m}\int d^3\vec{q}\,\frac{e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}}{\mathrm{q}}\frac{1}{\mathrm{q}+E_j-\left(E_i+E_f\right)/2}\tau_n^-\tau_m^-\,h_T(q^2)S_{nm}^q\middle|\psi_i\right\rangle$$

$$= \frac{R_A}{2\pi^2}\left\langle\psi_f\middle|\sum\nolimits_{n,m}\tau_n^-\tau_m^-\left[\vec{\sigma}_n.\vec{\sigma}_m\int\frac{d^3\vec{q}e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}h_T(q^2)}{\mathrm{q}\left[\mathrm{q}+E_j-\left(E_i+E_f\right)/2\right]} - 3\sum\nolimits_{i,j}\sigma_{ni}\sigma_{mj}\int\frac{d^3\vec{q}q_iq_je^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}h_T(q^2)}{\mathrm{q}^3\left[\mathrm{q}+E_j-\left(E_i+E_f\right)/2\right]}\right]\middle|\psi_i\right\rangle$$

$$= \frac{R_A}{2\pi^2}\langle\psi_f|\sum\nolimits_{n,m}\tau_n^-\tau_m^-[I_3+I_4]\,|\psi_i\rangle, \qquad (K.13)$$

where

$$I_3 = \vec{\sigma}_n.\vec{\sigma}_m\int\frac{d^3\vec{q}e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}h_T(q^2)}{\mathrm{q}\left[\mathrm{q}+E_j-\left(E_i+E_f\right)/2\right]}, \qquad (K.14)$$

and

$$I_4 = -3\sum_{i,j}\sigma_{ni}\sigma_{mj}\int\frac{d^3\vec{q}q_iq_je^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}h_T(q^2)}{\mathrm{q}^3\left[\mathrm{q}+E_j-\left(E_i+E_f\right)/2\right]}. \qquad (K.15)$$

Again using the results for the I_1-integral, in $(K.4)$, we can compute the I_3-integral:

$$I_3 = \vec{\sigma}_n.\vec{\sigma}_m\int\frac{d^3\vec{q}e^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}h_T(q^2)}{\mathrm{q}\left[\mathrm{q}+E_j-\left(E_i+E_f\right)/2\right]} = 4\pi\vec{\sigma}_n.\vec{\sigma}_m\int_0^\infty\frac{j_0(\mathrm{q}x_{nm})h_T(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q}+E_j-\left(E_i+E_f\right)/2}. \qquad (K.16)$$

On the other hand, using the results for the I_2-integral, in $(K.9)$, we can compute the I_4-integral:

$$I_4 = -3\sum_{i,j}\sigma_{ni}\sigma_{mj}\int\frac{d^3\vec{q}q_iq_je^{i\vec{q}.(\vec{x}_n-\vec{x}_m)}h_T(q^2)}{\mathrm{q}^3\left[\mathrm{q}+E_j-\left(E_i+E_f\right)/2\right]}$$

$$= 4\pi\left(\frac{3\vec{\sigma}_n.\vec{\sigma}_m}{x_{nm}} - \frac{3(\vec{\sigma}_n.\hat{x}_{nm})(\vec{\sigma}_m.\hat{x}_{nm})}{x_{nm}}\right)\int_0^\infty\frac{j_0'(\mathrm{q}x_{nm})h_T\left(q^2\right)d\mathrm{q}}{\mathrm{q}+E_j-\left(E_i+E_f\right)/2}$$

$$+4\pi\times 3(\vec{\sigma}_n.\hat{x}_{nm})(\vec{\sigma}_m.\hat{x}_{nm})\int_0^\infty\frac{j_0''(\mathrm{q}x_{nm})h_T\left(q^2\right)\mathrm{q}d\mathrm{q}}{\mathrm{q}+E_j-\left(E_i+E_f\right)/2}, \qquad (K.17)$$

where

$$\hat{x}_{nm} = \frac{\vec{x}_n - \vec{x}_m}{|\vec{x}_n - \vec{x}_m|}.$$

Adding $(K.16)$ and $(K.17)$, we have

$$I_3 + I_4 = 4\pi\vec{\sigma}_n.\vec{\sigma}_m \int_0^\infty \left\{j_0(\mathrm{q}x_{nm}) + \frac{3j_0^{'}(\mathrm{q}x_{nm})}{\mathrm{q}x_{nm}}\right\} \frac{h_T(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2}$$

$$+4\pi \times 3(\vec{\sigma}_n.\hat{x}_{nm})(\vec{\sigma}_m.\hat{x}_{nm}) \int_0^\infty \left\{j_0^{''}(\mathrm{q}x_{nm}) - \frac{j_0^{'}(\mathrm{q}x_{nm})}{\mathrm{q}x_{nm}}\right\} \frac{h_T(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2} \,. \qquad (K.18)$$

Now using the relation

$$j_0''(\mathrm{q}x_{nm}) - \frac{j_0'(\mathrm{q}x_{nm})}{\mathrm{q}x_{nm}} = -\left\{j_0(\mathrm{q}x_{nm}) + \frac{3j_0'(\mathrm{q}x_{nm})}{\mathrm{q}x_{nm}}\right\} = j_2(\mathrm{q}x_{nm}),$$

where $j_2(\mathrm{q}x_{nm})$ is the spherical Bessel function defined by,

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) sin(x) - \frac{3}{x^2} cos(x),$$

we can write

$$I_3 + I_4 = 4\pi\{3(\vec{\sigma}_n.\hat{x}_{nm})(\vec{\sigma}_m.\hat{x}_{nm}) - \vec{\sigma}_n.\vec{\sigma}_m\} \int_0^\infty \frac{j_2(\mathrm{q}x_{nm})h_T(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2}$$

$$= 4\pi S_{nm} \int_0^\infty \frac{j_2(\mathrm{q}x_{nm})h_T(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2}\,, \qquad S_{nm} = 3(\vec{\sigma}_n.\hat{x}_{nm})(\vec{\sigma}_m.\hat{x}_{nm}) - (\vec{\sigma}_n.\vec{\sigma}_m), \qquad (K.19)$$

Substituting $(K.19)$ into $(K.13)$, tensor nuclear matrix element can be written as:

$$M_T^{0\nu}$$

$$= \left\langle \psi_f \left| \sum_{n,m} \tau_n^- \tau_m^- S_{nm} \frac{2R_A}{\pi} \int_0^\infty \frac{j_2(\mathrm{q}x_{nm})h_T(q^2)\mathrm{q}d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2} \right| \psi_i \right\rangle. \qquad (K.20)$$

Thus The expressions for the Gamow-Teller, $M_{GT}^{0\nu}$, Fermi, $M_{F}^{0\nu}$, and Tensor, $M_{T}^{0\nu}$, nuclear matrix elements derived in $(K.10)$, $(K.11)$ and $(K.12)$, respectively, can be written in a compact form, which is essentially given by (5.4.22):

$M_{\alpha}^{0\nu} = \langle \psi_f | O_\alpha | \psi_i \rangle, \alpha = \{GT, F, T\}$,

with

$$O_{GT} = \sum_{n,m} \tau_n^- \tau_m^- (\vec{\sigma}_n . \vec{\sigma}_m) H_{GT}(X_{nm}, E_j), O_F = \sum_{n,m} \tau_n^- \tau_m^- H_F(X_{nm}, E_j),$$

$$O_T = \sum_{n,m} \tau_n^- \tau_m^- S_{nm} H_T(X_{nm}, E_j),$$

where

$$S_{nm} = 3(\vec{\sigma}_n . \hat{x}_{nm})(\vec{\sigma}_m . \hat{x}_{nm}) - (\vec{\sigma}_n . \vec{\sigma}_m), \vec{x}_{nm} = \vec{x}_n - \vec{x}_m, x_{nm} = |\vec{x}_{nm}|, \hat{x}_{nm} = \vec{x}_{nm}/x_{nm}$$

and

$$H_\alpha(r, E_j) = \frac{2R_A}{\pi} \int_0^\infty \frac{f_\alpha(\mathrm{q}r) h_\alpha(q^2) \mathrm{q} d\mathrm{q}}{\mathrm{q} + E_j - (E_i + E_f)/2}, \qquad (5.4.22)$$

where, $f_{GT,F}(\mathrm{q}r) = j_0(\mathrm{q}r)$ and $f_T(\mathrm{q}r) = j_2(\mathrm{q}r)$ are spherical Bessel functions and $h_\alpha(q^2)$'s are the form factors corresponding to nuclear matrix elements, $M_{\alpha}^{0\nu}$, given by the equation (5.4.14).

APPENDIX L

CASIMIR'S TRICK AND TRACE THEOREMS

In order to calculate the decay rate, $\Gamma^{0\nu}$, or the half-life, $T_{1/2}^{0\nu}$, for the $0\nu\beta\beta$ decay process we need to calcuate, $\left|V_{fi}^{0\nu}\right|^2$, where, $V_{fi}^{0\nu}$, is the transition amplitude, given by the equation (5.4.23). From equation (5.4.23), it follows that:

$$V_{fi}^{0\nu} \propto \overline{u}(p_1, s_1)(1+\gamma^5)C\overline{u}^T(p_2, s_2). \qquad (L.1)$$

As a typical experiment starts out with a beam of particles whose spin orientations are random, and simply counts the number of particles scattered in a given direction, in this case, the relevant cross section is the average over all initial spin configurations and the sum over all final spin configurations. Thus,

$$\left|V_{fi}^{0\nu}\right|^2 \equiv \mathit{average\ over\ initial\ spins\ and\ sum\ over\ final\ spins}. \qquad (L.2)$$

In order to evaluate the sum over the spin configurations of the emitting electrons, i.e., in order to evaluate the following quantity,

$$\sum_{all\ spins} \left|\overline{u}(p_1, s_1)(1+\gamma^5)C\overline{u}^T(p_2, s_2)\right|^2, \qquad (L.3)$$

let's at first consider a quantity, G, of the generic form, given by

$$G \equiv [\overline{u}(a)\Gamma_1 v(b)][\overline{u}(a)\Gamma_2 v(b)]^*, \qquad (L.4)$$

where, a, and, b, stand for the appropriate momenta and spins, and, Γ_1, and, Γ_2, are 4×4 matrices. To begin with, we evaluate the complex conjugate, which is the same as the Hermitian conjugate, since the quantity in square brackets is a 1×1 matrix.

$$[\overline{u}(a)\Gamma_2 v(b)]^* = [\overline{u}(a)\Gamma_2 v(b)]^\dagger = v(b)^\dagger \Gamma_2{}^\dagger \overline{u}(a)^\dagger = v(b)^\dagger \Gamma_2{}^\dagger \{u(a)^\dagger \gamma^0\}^\dagger$$

$$= v(b)^\dagger \Gamma_2{}^\dagger \gamma^{0\dagger} u(a), \qquad (L.5)$$

since the adjoint spinor, $\overline{u}(a)$, is defined by , $\overline{u}(a) = u(a)^\dagger\gamma^0$ and $(AB)^\dagger = B^\dagger A^\dagger$, $(A^\dagger)^\dagger = A$, for any two arbitrary matrices, A and B. Now, $\gamma^{0\dagger} = \gamma^0$, and $(\gamma^0)^2 = 1$, so,

$$[\overline{u}(a)\Gamma_2 v(b)]^* = v(b)^\dagger \Gamma_2{}^\dagger \gamma^{0\dagger} u(a) = v(b)^\dagger \gamma^0\gamma^0 \Gamma_2{}^\dagger \gamma^0 u(a) = \overline{v}(b)\overline{\Gamma}_2 u(a), \qquad (L.6)$$

where, $\overline{\Gamma}_2 = \gamma^0\Gamma_2{}^\dagger\gamma^0$. Thus substituting the expression for $[\overline{u}(a)\Gamma_2 v(b)]^*$ from $(L.6)$ into $(L.4)$, we have,

$$G = [\overline{u}(a)\Gamma_1 v(b)]\left[\overline{v}(b)\overline{\Gamma}_2 u(a)\right]. \qquad (L.7)$$

We are now ready to sum over the spin orientations of particle b. Using the completeness relation of free Dirac spinors, $v(b)$, given by $(B.7)$ in appendix B:

$$\sum_{b\ spins} v(p_b, s_b)\overline{v}(p_b, s_b) = \not{p}_b - m_b\,, \qquad \not{p}_b \equiv p_b^\mu \gamma_\mu\,, \qquad (L.8)$$

where, p_b, s_b and m_b stand for the momenta, spins and mass of particle b, we have

$$\sum_{b\ spins} G = \overline{u}(a)\Gamma_1 \left\{ \sum_{b\ spins} v(p_b, s_b)\overline{v}(p_b, s_b) \right\} \overline{\Gamma}_2 u(a) = \overline{u}(a)\Gamma_1 \left(\not{p}_b - m_b \right) \overline{\Gamma}_2 u(a)$$

$$= \overline{u}(a) Q u(a), \qquad Q = \Gamma_1 \left(\not{p}_b - m_b \right) \overline{\Gamma}_2, \qquad (L.9)$$

Summing over the spin orientations of particle a, we have

$$\sum_{a\ spins} \sum_{b\ spins} G = \sum_{a\ spins} \overline{u}(p_a, s_a) Q u(p_a, s_a). \qquad (L.10)$$

Using the completeness relation of free Dirac spinor, $u(a)$, given by $(B.7)$

$$\sum_{a\ spins} u(p_a, s_a)\overline{u}(p_a, s_a) = \not{p}_a + m_a\,, \qquad (L.11)$$

where, p_a, s_a and m_a stand for the momenta, spins and mass of particle a and writing out the matrix multiplication explicitly, we have the following expression from $(L.10)$:

$$\sum_{a\ spins}\sum_{b\ spins} G = \sum_{a\ spins}\sum_{i,j=1}^{4} \overline{u}(p_a, s_a)_i Q_{ij} u(p_a, s_a)_j = \sum_{i,j=1}^{4} Q_{ij} \left\{ \sum_{a\ spins} u(p_a, s_a)\overline{u}(p_a, s_a) \right\}_{ji}$$

$$= \sum_{i,j=1}^{4} Q_{ij} \left(\not{p}_a + m_a \right)_{ji} = \sum_{i=1}^{4} \left[Q \left(\not{p}_a + m_a \right) \right]_{ii} = Tr\left[Q \left(\not{p}_a + m_a \right) \right], \qquad (L.12)$$

where, $'Tr'$ denotes the trace of a matrix. Substituting $(L.4)$ and the expression for Q from $(L.9)$ into $(L.12)$, we have the following 'Casimir's Trick'

$$\sum_{all\ spins} [\overline{u}(a)\Gamma_1 v(b)][\overline{u}(a)\Gamma_2 v(b)]^* = Tr\left[\Gamma_1 \left(\not{p}_b - m_b \right) \overline{\Gamma}_2 \left(\not{p}_a + m_a \right) \right]. \qquad (L.13)$$

Casimir's Trick reduces everything to an exercise in calculating the trace of some complicated product of Dirac γ matrices. This algebra is facilitated by a number of theorems listed below:

If A and B are any two matrices and α is any scalar, then the following theorems hold:

1. $Tr(A + B) = Tr(A) + Tr(B)$
2. $Tr(\alpha A) = \alpha Tr(A)$
3. $Tr(AB) = Tr(BA)$

From the property of the metric tensor, $g^{\mu\nu} = diag(1, -1, -1, -1)$, and the fundamental anti-commutation relation for the Dirac γ matrices, given by:

4. $g_{\mu\nu} g^{\mu\nu} = 4$,
5. $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$,

following 'contraction theorems' hold:

6. $\not{a}\not{b} + \not{b}\not{a} = 2a.b$

7. $\gamma_\mu \gamma^\mu = 4$

8. $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$

9. $\gamma_\mu \not{a}\, \gamma^\mu = -2 \not{a}$

10. $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$

11. $\gamma_\mu \not{a}\not{b}\, \gamma^\mu = 4(a.b)$

A collection of trace theorems also follows from (4) and (5), which is given below:

12. *The trace of the product of an odd number of gamma matrices is zero.*

13. $Tr(1) = 4$

14. $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

15. $Tr(\not{a}\not{b}) = 4(a.b)$

Since, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, is the product of an even number of gamma matrices, it follows from trace theorem, (12), that

16. *The trace of the product of an odd number of gamma matrices with* γ^5 *is zero.*

When γ^5 is multiplied by even number of gamma matrices, the following theorems hold

17. $Tr(\gamma^5) = 0$

18. $Tr(\gamma^5 \gamma^\mu \gamma^\nu) = 0$

19. $Tr(\gamma^5 \not{a}\not{b}) = 0$

Now let's evaluate the quantity given by $(L.3)$. Since for Dirac spinor, $C\overline{u}^T(p,s) = v(p,s)$.

Using Casimir's Trick from $(L.13)$, we have

$$\sum_{all\ spins} \left|\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)\right|^2 = \sum_{all\ spins} |\overline{u}(p_1,s_1)(1+\gamma^5)v(p_2,s_2)|^2$$

$$= \sum_{all\ spins} [\overline{u}(p_1,s_1)(1+\gamma^5)v(p_2,s_2)][\overline{u}(p_1,s_1)(1+\gamma^5)v(p_2,s_2)]^*$$

$$= Tr\left[(1+\gamma^5)\left(\not{p}_2 - m_e\right)(1-\gamma^5)\left(\not{p}_1 + m_e\right)\right], \quad (L.14)$$

where, m_e is the mass of electron, $\Gamma_1 = \Gamma_2 = 1+\gamma^5$ and $\overline{\Gamma}_2 = \gamma^0{\Gamma_2}^\dagger\gamma^0 = \gamma^0(1+\gamma^5)^\dagger\gamma^0 = \gamma^0(1+\gamma^5)\gamma^0 = (\gamma^0)^2 + \gamma^0\gamma^5\gamma^0 = 1 - \gamma^0\gamma^0\gamma^5 = 1-\gamma^5$, since, $\gamma^{5\dagger} = \gamma^5$, $(\gamma^0)^2 = 1$ and $\gamma^5\gamma^0 = -\gamma^0\gamma^5$.

Using trace theorems (1) and (2), we have

$$Tr\left[(1+\gamma^5)\left(\not{p}_2 - m_e\right)(1-\gamma^5)\left(\not{p}_1 + m_e\right)\right]$$

$$= Tr\left[\left\{\not{p}_2 + \gamma^5\not{p}_2 - m_e(1+\gamma^5)\right\}\left\{\not{p}_1 - \gamma^5\not{p}_1 + m_e(1-\gamma^5)\right\}\right]$$

$$= Tr\left[\not{p}_2\not{p}_1 - \gamma^5\not{p}_2\gamma^5\not{p}_1\right] + Tr\left[\gamma^5\not{p}_2\not{p}_1 - \not{p}_2\gamma^5\not{p}_1\right] - m_e^2 Tr[1-(\gamma^5)^2]$$

$$+ m_e Tr\left[\not{p}_2 - \not{p}_1 + \gamma^5\not{p}_2 - \not{p}_2\gamma^5 + (\gamma^5)^2\not{p}_1 - \gamma^5\not{p}_2\gamma^5\right]. \quad (L.15)$$

Substituting $(L.15)$ into $(L.14)$,we have

$$\sum_{all\ spins} \left|\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)\right|^2$$

$$= Tr\left[\not{p}_2\not{p}_1 - \gamma^5\not{p}_2\gamma^5\not{p}_1\right] + Tr\left[\gamma^5\not{p}_2\not{p}_1 - \not{p}_2\gamma^5\not{p}_1\right] - m_e^2 Tr[1-(\gamma^5)^2]$$

$$+ m_e Tr\left[\not{p}_2 - \not{p}_1 + \gamma^5\not{p}_2 - \not{p}_2\gamma^5 + (\gamma^5)^2\not{p}_1 - \gamma^5\not{p}_2\gamma^5\right]. \quad (L.16)$$

Since, $(\gamma^5)^2 = 1$ and $\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$, it follows from trace theorems (1) and (15) that

$$Tr\left[\not p_2 \not p_1 - \gamma^5 \not p_2 \gamma^5 \not p_1\right] = Tr\left[\not p_2 \not p_1 + (\gamma^5)^2 \not p_2 \not p_1\right] = 2Tr\left[\not p_2 \not p_1\right] = 2 \times 4p_1.p_2$$

$$= 8p_1.p_2 = 8(E_1E_2 - \vec{p}_1.\vec{p}_2) = 8(E_1E_2 - |\vec{p}_1||\vec{p}_2|cos\theta) = 8E_1E_2\left(1 - \frac{|\vec{p}_1||\vec{p}_2|}{E_1E_2}cos\theta\right)$$

$$= 8E_1E_2(1 - \alpha cos\theta),$$

with

$$\alpha = \frac{|\vec{p}_1||\vec{p}_2|}{E_1E_2}, \qquad (L.17)$$

where, p_1, p_2 are the four-momenta of the emitting electrons, $\vec{p}_1$, $\vec{p}_2$ and E_1, E_2 are the momenta and energies of the electrons , respectively, and θ is the angle between the emitting electrons of the $0\nu\beta\beta$ beta decay process.

From trace theorem (19) it follows that,

$$Tr\left[\gamma^5 \not p_2 \not p_1 - \not p_2 \gamma^5 \not p_1\right] = 0. \qquad (L.18)$$

$(\gamma^5)^2 = 1$, implies that,

$$m_e^2 Tr[1 - (\gamma^5)^2] = 0. \qquad (L.19)$$

From trace theorems (12) and (16), it follows that:

$$m_e Tr\left[\not p_2 - \not p_1 + \gamma^5 \not p_2 - \not p_2 \gamma^5 + (\gamma^5)^2 \not p_1 - \gamma^5 \not p_2 \gamma^5\right] = 0. \qquad (L.20)$$

Substituting $(L.17), (L.18), (L.19)$ and $(L.20)$ into $(L.16)$ we finally have have:

$$\sum_{all\ spins} \left|\overline{u}(p_1, s_1)(1 + \gamma^5)C\overline{u}^T(p_2, s_2)\right|^2 = 8E_1E_2(1 - \alpha cos\theta), \qquad \alpha = \frac{|\vec{p}_1||\vec{p}_2|}{E_1E_2}. \qquad (L.21)$$

APPENDIX M

CALCULATION OF THE PHASE SPACE ELEMENTS

The differential decay rate, $d\Gamma^{0\nu}$ for the $0\nu\beta\beta$ decay process is given by the following equation:

$$d\Gamma^{0\nu} = \left|V_{fi}^{0\nu}\right|^2 \frac{1}{2} \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} 2\pi\delta\left(E_1 + E_2 + E_f - E_i\right), \qquad (O.2.3)$$

where, $V_{if}^{0\nu}$, is the transition amplitude given by the equation $(5.4.23)$, $\vec{p}_1$, $\vec{p}_2$ and E_1, E_2 are the momenta and energies of emitting electrons respectively, E_i and E_f are the energies of the initial and final nuclear states of the of the $0\nu\beta\beta$ beta decay process. The momentum elements, $d^3\vec{p}_1$ and $d^3\vec{p}_2$ can be written as:

$$d^3\vec{p}_1 = 2\pi p_1'^2 dp_1' sin\theta d\theta, \qquad d^3\vec{p}_2 = 4\pi p_2'^2 dp_2', \qquad p_k' = |\vec{p}_k|, \qquad k = 1,2, \qquad (M.1)$$

with the relation between the energy and the momentum for electrons, given by:

$$E_k^2 = p_k'^2 + m_e^2, \qquad (M.2)$$

where, θ is the angle between the emitting electrons and, m_e is the mass of electrons.

Differentiating the equation $(M.2)$, we get:

$$p_k' dp_k' = E_k dE_k. \qquad (M.3)$$

Substituting equation $(M.3)$ into equation $(M.1)$, we have:

$$d^3\vec{p}_1 = 2\pi p_1' E_1 dE_1 sin\theta d\theta, \qquad d^3\vec{p}_2 = 4\pi p_2' E_2 dE_2. \qquad (M.4)$$

Substituting equation $(M.4)$ into equation $(O.2.3)$, we can write:

$$d\Gamma^{0\nu} = \frac{\left|V_{fi}^{0\nu}\right|^2}{4(2\pi)^3} p_1' p_2' dE_1 dE_2 sin\theta d\theta \delta\left(E_1 + E_2 + E_f - E_i\right). \qquad (M.5)$$

Now from the transition amplitude, $V_{if}^{0\nu}$, for the $0\nu\beta\beta$ beta decay is given by:

$$V_{fi}^{0\nu} = -\frac{g_A^2 m_e}{2\pi}\left(\frac{G_F}{\sqrt{2}}\right)^2 \frac{1}{R_A}\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)M^{0\nu}\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right), \qquad (5.4.23)$$

where , $M^{0\nu}$, is the nuclear matrix element, we get

$$\left|V_{fi}^{0\nu}\right|^2 = \frac{2m_e^4 G_F^4}{(2\pi)^2}\frac{g_A^4}{r_A^2}E_1E_2(1-\alpha cos\theta)|M^{0\nu}|^2\left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2, \qquad (M.6)$$

by summing over the spins of the emitting electrons using Casimir's trick described in the appendix L and substituting the following expression:

$$\sum_{all\ spins}\left|\overline{u}(p_1,s_1)(1+\gamma^5)C\overline{u}^T(p_2,s_2)\right|^2 = 8E_1E_2(1-\alpha cos\theta), \qquad \alpha = \frac{|\vec{p}_1||\vec{p}_2|}{E_1E_2}, \qquad (L.21)$$

into the equation (5.4.23). Here, $r_A = m_e R_A$, $R_A = 1.2A^{1/3}$fm, R_A is the radius of the daughter nuclei of mass number, A.

Substituting the equation $(M.6)$ into the equation$(M.5)$, we have the following formula for the differential decay rate of the the $0\nu\beta\beta$ beta decay process:

$$d\Gamma^{0\nu} = \frac{m_e^4 G_F^4}{2(2\pi)^5}\frac{g_A^4}{r_A^2}p_1'p_2'E_1dE_1E_2dE_2\delta\left(E_1+E_2+E_f-E_i\right)$$

$$\times(1-\alpha cos\theta)sin\theta d\theta|M^{0\nu}|^2\left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2. \qquad (M.7)$$

Integrating over the angles between the emitting electrons, θ,

$$\int_0^\pi(1-\alpha cos\theta)sin\theta\, d\theta = \int_0^\pi sin\theta\, d\theta = 2, \qquad (M.8)$$

and over the energy of electron, E_2,

$$\int E_2\delta\left(E_1+E_2+E_f-E_i\right)dE_2 = E_i - E_f - E_1 = E_2, \qquad (M.9)$$

with the principal of conservation of energy, $E_i = E_1 + E_2 + E_f$, we have arrived the formula for the differential decay rate, $d\Gamma^{0\nu}$, given by the equation (5.5.2):

$$d\Gamma^{0\nu} = \frac{m_e^4 G_F^4}{(2\pi)^5}\frac{g_A^4}{r_A^2} p_1' p_2' E_1 E_2 dE_1 |M^{0\nu}|^2 \left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2. \tag{5.5.2}$$

Introducing the momenta, $\tilde{p}_1$ and $\tilde{p}_2$ and energies, ε_1 and ε_2 of electrons in the units of electron mass,m_e, defined by:

$$\varepsilon_k = \frac{E_k}{m_e}, \qquad \tilde{p}_k = \frac{p_k'}{m_e} = \frac{|\vec{p}_k|}{m_e} = \sqrt{\varepsilon_k^2 - 1}, \qquad k = 1,2, \tag{M.10}$$

and including the relativistic Fermi factor, $F_0(Z_f, \varepsilon_k)$, of coulomb corrections, given by (5.5.3) the following expression:

$$F_0(Z_f, \varepsilon_k) = \frac{4}{[\Gamma(2\gamma_1 + 1)]^2}(2p_k' R_A)^{2(\gamma_1 - 1)} |\Gamma(\gamma_1 + i y_k)|^2 e^{\pi y_k}, \qquad k = 1,2,$$

with

$$\gamma_1 = \sqrt{1 - (\alpha Z_f)^2}, \qquad y_k = \alpha Z_f \varepsilon_k / p_k', \qquad Z_f = Z_i + 2, \tag{5.5.3}$$

where, $\alpha = 1/137$, is the fine-structure constant, $\Gamma(x)$, is the gamma function and Z_i and Z_f are the atomic numbers of the initial and final nucleus respectively, we can rewrite the differential decay rate, $d\Gamma^{0\nu}$ of the of the $0\nu\beta\beta$ beta decay process in the following form:

$$d\Gamma^{0\nu} = \frac{m_e^9 G_F^4}{(2\pi)^5}\frac{g_A^4}{r_A^2} F_0(Z_f, \varepsilon_1) F_0(Z_f, \varepsilon_2) \tilde{p}_1 \tilde{p}_2 \varepsilon_1 \varepsilon_2 d\varepsilon_1 |M^{0\nu}|^2 \left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2. \tag{M.11}$$

Integrating over the energy of electron,ε_1, we have arrived the following formula for the decay rate, $\Gamma^{0\nu}$, given by the equation, (5.5.4)

$$\Gamma^{0\nu} = \int_1^{T+1} d\Gamma^{0\nu} = \frac{m_e^9 G_F^4}{(2\pi)^5}\frac{g_A^4}{r_A^2} \int_1^{T+1} \left[b_1^{\beta\beta} F_0(Z_f, \varepsilon_1) F_0(Z_f, \varepsilon_2) \tilde{p}_1 \tilde{p}_2 \varepsilon_1 \varepsilon_2 d\varepsilon_1\right] |M^{0\nu}|^2 \left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2, \tag{5.5.4}$$

where $b_1^{\beta\beta} = 1, \varepsilon_2 = T + 2 - \varepsilon_1$ and T is the Q-value of the of the $0\nu\beta\beta$ beta decay process in the units of electron mass, m_e.

Change of limit for the energy integral to calculate the decay rate, $\Gamma^{0\nu}$.

	$lower\ limit\ of\ energy$	$upper\ limit\ of\ energy$
E_1	m_e	$E_i - E_f - m_e = Q + m_e$
$\varepsilon_1 = E_1/m_e$	1	$Q/m_e + 1 = T + 1$

Half-life, $T_{1/2}^{0\nu}$, of the the $0\nu\beta\beta$ beta decay process is related to the decay rate, $\Gamma^{0\nu}$, by the following formula:

$$\left[T_{1/2}^{0\nu}\right]^{-1} = \frac{\Gamma^{0\nu}}{ln2}. \quad (5.5.5)$$

Substituting the equation (5.5.4) into the equation (5.5.5), we have finally arrived the following formula for the half-life, $T_{1/2}^{0\nu}$, of the the $0\nu\beta\beta$ beta decay process:

$$\left[T_{1/2}^{0\nu}\right]^{-1} = G_1^{0\nu}|M^{0\nu}|^2 \left|\left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)\right|^2, \quad (5.5.6)$$

where phase space integral, $G_1^{0\nu}$, is given by:

$$G_1^{0\nu} = \frac{g^{0\nu}}{r_A^2}\int_1^{T+1} b_1^{\beta\beta}\, F_0(Z_f, \varepsilon_1)F_0(Z_f, \varepsilon_2)\tilde{p}_1\tilde{p}_2\varepsilon_1\varepsilon_2 d\varepsilon_1,$$

with

$$g^{0\nu} = (m_e^9 G_F^4 g_A^4)/(32\pi^5 ln2) = 2.80 \times 10^{-22} g_A^4\, yr^{-1}. \quad (5.5.7)$$

APPENDIX N

WICK'S THEOREM AND ITS APPLICATION TO THE EXPANSION OF S OPERATOR

N.1. Wick's theorem

In section (4.3), we have deduced a series expansion of the S operator, known as Dyson expansion, in powers of the interaction Hamiltonian density, $\mathcal{H}_I^I(x)$. Thus in principle perturbation theory can be carried out at arbitrary higher orders. When attempting to do so one is confronted with the technical problem to evaluate time ordered products of the type $T[\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)\dots\mathcal{H}_I^I(x_n)]$. The interaction Hamiltonian density, $\mathcal{H}_I^I(x)$ will consist of products of field operators describing interacting quantum fields. As one can imagine, the evaluation of higher order T products can be quite cumbersome to handle mathematically, because one cannot keep the same order of operators throughout the integration over time. In principal, this is not a difficult calculation: by manually applying the canonical commutation relations, one can reorder the field operators until one is left with a simple result. This can be a tedious task, especially at higher orders of perturbation theory. Fortunately, there is a very elegant and efficient tool that can be used to evaluate systematically any complicated time ordered product into non-time ordered form, where the order of operators does not change during integration over time. This is the basis of Wick's theorem, developed by Gian-Carlo Wick.

Wick's theorem is extensively used in quantum field theory to reduce any arbitrary time ordered product of creation and annihilation operators as a sum of normal ordered products of those operators using 'contractions' and 'normal ordering'. The definitions of these terms are given below:

(i) Contraction: Let $A(x)$ and $B(x)$ represent two generic field operators, which can be written as a sum of creation and annihilation operators:

$$A(x) = A^{(+)}(x) + A^{(-)}(x), \qquad B(x) = B^{(+)}(x) + B^{(-)}(x), \qquad (N.1.1)$$

where '$(+)$' and '$(-)$' represents 'creation' and 'annihilation' respectively. A contraction, $\overbrace{A(x_1)B(x_2)}$ of the operators $A(x)$ and $B(x)$ is defined as:

$$\overbrace{A(x_1)B(x_2)} = \begin{cases} \left[A^{(+)}(x_1), B^{(-)}(x_2)\right]_{\mp} & if \quad t_1 > t_2 \\ \pm\left[B^{(+)}(x_2), A^{(-)}(x_1)\right]_{\mp} & if \quad t_2 > t_1 \end{cases}, \qquad (N.1.2)$$

where the plus sign subscript implies anti-commutation, which should be used if both $A(x)$ and $B(x)$ represent fermionic fields, on the other hand, the minus sign subscript implies commutation, which should be used if both $A(x)$ and $B(x)$ represent bosonic fields. Since we are only dealing with fermionic fields (fields of electrons and Majorana neutrinos) for the $0\nu\beta\beta$ decay process, we always refer to the anti-commutation and avoid the plus sign as a subscript.

(ii) Normal ordered product: Normal ordering of operators refers to the fact that the operators should be arranged in a way that all annihilation operators are placed to the right of all creation operators. Thus if $A(x)$ and $B(x)$ represent two generic fermionic field operators, the normal ordered product $N[A(x_1)B(x_2)]$ is given by;

$$N[A(x_1)B(x_2)] = A^{(-)}(x_1)B^{(-)}(x_2) + A^{(-)}(x_1)B^{(+)}(x_2)$$

$$+A^{(+)}(x_1)B^{(+)}(x_2) - B^{(-)}(x_2)A^{(+)}(x_1). \qquad (N.1.3)$$

Note that the minus sign in the last term in $(N.1.3)$, arises due to the anticommutation property of the fermionic fields. With these definitions given above, we can now formally state the Wick's theorem:

The time ordered product of a set of operators can be decomposed into the sum of the corresponding contracted normal ordered products. All contractions of pairs of operators that possibly can arise enter this sum. Thus if $A, B, C, \ldots \ldots \ldots X, Y, Z$ represent the field operators, we can write the Wicks theorem as follows:

$$T[ABC \ldots XYZ] = N[ABC \ldots XYZ]$$

$$+N\left[\overbrace{AB}\,C \ldots XYZ\right] + N\left[\overbrace{ABC} \ldots XYZ\right] + \ \ldots \ldots \ldots + N\left[ABC \ldots X\,\overbrace{YZ}\right]$$

$$+N\left[\overbrace{AB}\,\overbrace{CD} \ldots XYZ\right] + \ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + N\left[ABC \ldots \overbrace{WX}\,\overbrace{YZ}\right]$$

+ sum over all triply contracted normal ordered terms with non-equal times contractions

+ sum over all normal ordered terms with higher contractions. $(N.1.4)$

N.2. Application of Wick's theorem to the expansion of S operator

In this section, using Wick's theorem given by $(N.1.4)$, we will evaluate the following quantity, which appears in the expression for the second order perturbative matrix element, $\langle f|S^{(2)}|i\rangle^{0\nu}$ of $0\nu\beta\beta$ decay process:

$$\langle p_2, s_2; p_1, s_1|T\left[\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right]|0\rangle, \qquad (N.2.1)$$

where, $e(x_1)$, $e(x_2)$ and $\nu_{eL}(x_1), \nu_{eL}(x_2)$ represents the field operators for the electrons and left-handed electron-neutrinos, respectively. $|0\rangle$ and $|p_2, s_2; p_1, s_1\rangle$ represents the vacuum state and the two-particle state for electrons with four-momentum, p_1, p_2 and spin projections, s_1, s_2 ,respectively.

Applying Wick's theorem $(N.1.4)$, we can write:

$$T\left[\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right] = N\left[\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right]$$

$$+N\left[\overbrace{\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)}\,(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right] + N\left[\overbrace{\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)}\right]$$

$$+N\left[\overline{e}_L(x_1)\gamma^{\mu}\,\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}\,(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right]+N\left[\overline{e}_L(x_1)\gamma^{\mu}\,\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)}\right]$$

+ sum over all triply contracted normal ordered terms with non-equal times contractions

+ sum over all normal ordered terms with higher contractions.

$$=N\left[\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right]+N\left[\overline{e}_L(x_1)\gamma^{\mu}\,\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}\,(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right],\qquad (N.2.2)$$

In the last line we used the fact that, for the $0\nu\beta\beta$ decay process, at the final stage of decay, we have two electrons and no neutrinos. Thus both electrons are created but neutrinos are created and absorbed, therefore the only contraction that survives is the contraction of the electron-neutrino fields, $\nu_{eL}(x_1)$ and $\nu_{eL}(x_2)$.

Substituting $(N.2.2)$ into $(N.2.1)$, we have

$$\langle p_2,s_2;p_1,s_1|T[\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)]|0\rangle$$

$$=\langle p_2,s_2;p_1,s_1|N[\overline{e}_L(x_1)\gamma^{\mu}\nu_{eL}(x_1)\nu_{eL}^T(x_2)(\gamma^{\nu})^T\overline{e}_L^T(x_2)]|0\rangle$$

$$+\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}_L(x_1)\gamma^{\mu}\,\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}\,(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right]\middle|0\right\rangle$$

$$=\left\langle p_2,s_2;p_1,s_1\middle|N\left[\overline{e}_L(x_1)\gamma^{\mu}\,\overbrace{\nu_{eL}(x_1)\nu_{eL}^T(x_2)}\,(\gamma^{\nu})^T\overline{e}_L^T(x_2)\right]\middle|0\right\rangle,\qquad (N.2.3)$$

Here we have used the fact that since the initial state is vacuum, $|0\rangle$, the expectation value of the normal order of the first term in the second line of $(N.2.3)$ is zero.

APPENDIX O

PERTURBATION THEORY AND DECAY RATE

This section is devoted to the formal development of the perturbation theory [100], which led us to the decay rate $\Gamma^{0\nu}$ for the $0\nu\beta\beta$ decay process. For a $0\nu\beta\beta$ decay, a parent nuclei decays to a daughter nuclei with the emission of two electrons, where the states, $|i\rangle$ and $|f\rangle$, corresponding to the initial and final nuclei are assumed to be nonrelativistic, i.e., the hadronic current, $J_L^{\mu\dagger}(\vec{x})$ is nonrelativistic. Thus a nonrelativistic treatment of the perturbation theory is necessary to derive the decay rate $\Gamma^{0\nu}$ formula for the $0\nu\beta\beta$ decay process. However, the states corresponding to the electrons are described by the relativistic quantum fields. We will include the relativistic effects of the leptonic fields, later through the density of states by introducing the covariant normalization of the free Dirac fields of electrons.

O.1. Nonrelativistic perturbation theory

Assume a system is described by a nonrelativistic Hamiltonian, H:

$$H\psi = i\frac{\partial\psi}{\partial t}, \qquad (O.1.1)$$

and that H has the following form:

$$H = H_0 + V(\vec{x}, t), \qquad (O.1.2)$$

where H_0 is the time independent unperturbed Hamiltonian, for which the eigenfunctions, $\psi_n(\vec{x})$ are known, and $V(\vec{x}, t)$ is the time-dependent perturbation. The eigenfunctions, $\psi_n(\vec{x})$ satisfy the following conditions:

$$H_0\psi_n(\vec{x}) = E_n\psi_n(\vec{x}), \qquad \int d^3\vec{x}\,\psi_m^*(\vec{x})\psi_n(\vec{x}) = \delta_{mn}, \qquad (O.1.3)$$

where E_n is the energy associated with the eigenfunctions, $\psi_n(\vec{x})$. For simplicity, the eigenfunctions are normalized to one particle in a box of volume, V.

The basic strategy is to express the solution of $(O.1.1)$ as a sum over the eigenstates of H_0 with time-dependent coefficients:

$$\psi(\vec{x},t) = \sum_n a_n(t)\,\psi_n(\vec{x})e^{-iE_nt}. \tag{O.1.4}$$

To find the unknown coefficients, $a_n(t)$, we substitute $(O.1.4)$ into $(O.1.1)$ and obtain:

$$i\sum_n \frac{da_n(t)}{dt}\psi_n(\vec{x})e^{-iE_nt} = \sum_n a_n(t)V(\vec{x},t)\psi_n(\vec{x})e^{-iE_nt}. \tag{O.1.5}$$

Multiplying by $\psi_f^*(\vec{x})$, integrating over the volume and using the orthonormality condition $(O.1.3)$ lead to the following coupled linear differential equations for the coefficients, $a_n(t)$:

$$\frac{da_f(t)}{dt} = -i\sum_n a_n(t)\left[\int d^3\vec{x}\psi_f^*(\vec{x})V(\vec{x},t)\psi_n(\vec{x})\right]e^{i(E_f-E_n)t}. \tag{O.1.6}$$

Suppose that before the interaction potential, $V(\vec{x},t)$, acts, the particle is in the eigenstate, i of the unperturbed Hamiltonian , H_0, that is, at time, $t = -T/2$:

$$a_i(-T/2) = 1, \qquad a_n(-T/2) = 0, \qquad n \neq i, \tag{O.1.7}$$

and

$$\frac{da_f(t)}{dt} = -i\int d^3\vec{x}\psi_f^*(\vec{x})V(\vec{x},t)\psi_i(\vec{x})e^{i(E_f-E_i)t}. \tag{O.1.8}$$

Now, provided that the interaction potential, $V(\vec{x},t)$, is small and transient, we can, as a first approximation, assume that these initial conditions $(O.1.7)$ remain valid at all times. Thus, integrating $(O.1.8)$, we obtain:

$$a_f(t) = -i\int_{-T/2}^{t} dt'\int d^3\vec{x}\psi_f^*(\vec{x})V(\vec{x},t)\psi_i(\vec{x})e^{i(E_f-E_i)t'}, \tag{O.1.9}$$

and, in particular, at time, $t = +T/2$, after the interaction potential, $V(\vec{x}, t)$ has ceased, we have:

$$T_{fi} \equiv a_f(T/2) = -i \int_{-T/2}^{+T/2} dt \int d^3\vec{x} \left[\psi_f(\vec{x}) e^{-iE_f t}\right]^* V(\vec{x}, t) \left[\psi_i(\vec{x}) e^{-iE_i t}\right]. \qquad (O.1.10)$$

The expression given by $(O.1.10)$ can be written in the following covariant form:

$$T_{fi} = -i \int d^4x \psi_f^*(x) V(x) \psi_i(x). \qquad (O.1.11)$$

From the expression for T_{fi}, given by $(O.1.11)$, it is possible to interpret $|T_{fi}|^2$ as the probability that a particle undergoes a transition from an initial state i to a final state f. In particular, when the interaction potential, $V(\vec{x}, t)$ is time independent, that is, $V(\vec{x}, t) = V(\vec{x})$, then, $(O.1.11)$ can be written as:

$$T_{fi} = -iV_{fi} \int_{-\infty}^{\infty} dt e^{i(E_f - E_i)t} = -i2\pi\delta(E_f - E_i) V_{fi}, \qquad (O.1.12)$$

with

$$V_{fi} \equiv \int d^3\vec{x} \psi_f^*(\vec{x}) V(\vec{x}) \psi_i(\vec{x}). \qquad (O.1.13)$$

V_{fi} is often called the transition amplitude for the transition from an initial state i to a final state f. The Dirac δ-function in $(O.1.12)$ expresses the fact that the energy of the particle is conserved in the transition $i \to f$. By the uncertainty principle, this means that, there is an infinite time separation between the initial states i and the final state f, and therefore, $|T_{fi}|^2$ is not a meaningful quantity. We define instead a transition probability per unit time, W_{if}, given by,

$$W_{fi} = \lim_{T \to 0} \frac{|T_{fi}|^2}{T}. \qquad (O.1.14)$$

Squaring $(O.1.12)$ and substituting into $(O.1.14)$, we have:

$$W_{fi} = \lim_{T\to 0}\left|V_{fi}\right|^2 2\pi\delta\left(E_f - E_i\right)\frac{1}{T}\int_{-T/2}^{+T/2} dte^{i(E_f - E_i)t} = \lim_{T\to 0}\left|V_{fi}\right|^2 2\pi\delta\left(E_f - E_i\right)\frac{1}{T}\int_{-T/2}^{+T/2} dt$$

$$= \left|V_{fi}\right|^2 2\pi\delta\left(E_f - E_i\right). \qquad (O.1.15)$$

In particle and nuclear physics, we usually deal with a situation where we start with a specified initial state and end up in one of a set of final states. The number of final states, dn_f in the energy interval E_f to $E_f + dE_f$ is given by:

$$dn_f = \rho\left(E_f\right)dE_f, \qquad (O.1.16)$$

where $\rho\left(E_f\right)$ is the density of final states. Thus from $(O.1.15)$ and $(O.1.16)$, the differential transition rate can be written as:

$$dW_{fi} = \left|V_{fi}\right|^2 dn_f 2\pi\delta\left(E_f - E_i\right) = \left|V_{fi}\right|^2 \rho\left(E_f\right)dE_f 2\pi\delta\left(E_f - E_i\right). \qquad (O.1.17)$$

Integrating $(O.1.17)$ over the density of final states, $\rho\left(E_f\right)$ and imposing the energy conservation, we obtain an expression for the transition rate, W_{fi}, known as Fermi's Golden Rule:

$$W_{fi} = \int \left|V_{fi}\right|^2 \rho\left(E_f\right)dE_f 2\pi\delta\left(E_f - E_i\right) = 2\pi\left|V_{fi}\right|^2 \rho(E_i). \qquad (O.1.18)$$

O.2. Decay rate for neutrinoless double beta decay process

Fermi's Golden Rule is used for the derivation of the formula for decay rate. One of the essential ingredients to compute the decay rate is the density of final states, $\rho\left(E_f\right)$ of the decaying particles coming from the phase space contributions of final particles. For free particles, we choose the covariant normalization of the fields describing the particles, in a way that we have $2E$ particles per unit volume, where E is the energy of a decaying particle. Since for a single particle the number of final states, dn_f in a volume, V with momenta elements, $d^3\vec{p}_f$ to be:

$$dn_f = \frac{V d^3\vec{p}_f}{(2\pi)^3 2E}, \qquad (O.2.1)$$

the relativistic generalization of the differential decay rate, $d\Gamma$ for the decays $A \rightarrow 1 + 2 + \cdots\cdots\cdots + n$ into momentum elements $d^3\vec{p}_1$, $d^3\vec{p}_2, \dots\dots, d^3\vec{p}_n$ of the final state particles is given by:

$$d\Gamma = \frac{1}{2E_A}\left|M_{fi}\right|^2 \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \dots\dots \frac{d^3\vec{p}_n}{(2\pi)^3 2E_n} (2\pi)^4 \delta^{(4)}(p_A - p_1 + \ \dots . + p_n), \qquad (O.2.2)$$

where $2E_A$ is the number of decaying particles per unit volume, p_A is the four-momenta of the parent nuclei, $p_1, p_2, \dots p_n$ and $E_1, E_2, \dots E_n$ are the four-momenta and energies of the decaying particles, respectively and M_{fi} is the invariant amplitude which is computed from the relevant Feynman diagram.

For the case of the $0\nu\beta\beta$ decay, however, we use the non-relativistic Fermi's Golden Rule $(O.1.18)$ to calculate the decay rate, $\Gamma^{0\nu}$, since we assume the impulse approximation both for the parent and daughter nuclei but the phase space contributions coming from the free electrons will still be treated as relativistic. Thus the differential decay rate, $d\Gamma^{0\nu}$ for the $0\nu\beta\beta$ decay is given by:

$$d\Gamma^{0\nu} = \left|V_{fi}^{0\nu}\right|^2 \frac{1}{2} \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} 2\pi\delta\left(E_1 + E_2 + E_f - E_i\right), \qquad (O.2.3)$$

where, $V_{if}^{0\nu}$, is the transition amplitude for the $0\nu\beta\beta$ decay process given by the expression $(4.4.23)$, $\vec{p}_1$, $\vec{p}_2$ and E_1, E_2 are the momenta and energies of emitting electrons respectively, E_i and E_f are the energies of the initial and final nuclear states of the $0\nu\beta\beta$ beta decay process.

Note that the factor $1/2$ in $(O.2.3)$ arises because electrons are identical particles. Thus for the two final state electrons, we have to divide the differential decay rate, $d\Gamma^{0\nu}$ by the symmetry factor, $S_{sym} = 2! = 2$. The total the decay rate, $\Gamma^{0\nu}$ can be calculated by integrating over all outgoing momenta of electrons:

$$\Gamma^{0\nu} = \int d\Gamma^{0\nu}. \quad (O.2.4)$$

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